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Spaces of quasi-exponentials and representations of $\mathfrak{gl}_N$

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Abstract

We consider the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$, the weight subspace of weight $\boldsymbol{\lambda}$ of the tensor product of k polynomial irreducible $\mathfrak{gl}_N$-modules with highest weights $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}$, respectively. The Bethe algebra depends on N complex numbers $\boldsymbol{K} = (K_1, \ldots, K_N)$. Under the assumption that $K_1, \ldots, K_N$ are distinct, we prove that the image of $\mathcal{B}_{\boldsymbol{K}}$ in $\operatorname{End}\left(\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}\right)$ is isomorphic to the algebra of functions on the intersection of suitable Schubert cycles in the Grassmannian of N-dimensional spaces of quasi-exponentials with exponents $\boldsymbol{K}$. We also prove that the $\mathcal{B}_{\boldsymbol{K}}$-module $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$ is isomorphic to the coregular representation of that algebra of functions. We present a Bethe ansatz construction identifying the eigenvectors of the Bethe algebra with points of that intersection of Schubert cycles.

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1. Introduction

It has been proved recently in [MTV6] that the eigenvectors of the Bethe algebra of the $\mathfrak{gl}_N$ Gaudin model are in a bijective correspondence with Nth-order Fuchsian differential operators with polynomial kernel and prescribed singularities. In this paper, we construct a variant of this correspondence.

The Bethe algebra considered in [T, MTV6] admits a deformation $\mathcal{B}_{\boldsymbol{K}}$ depending on N complex parameters $\boldsymbol{K} = (K_1, \ldots, K_N)$ (see [CT, MTV1]). Under the assumption

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that $K_1, \ldots, K_N$ are distinct, we consider the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ acting on $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$, the weight subspace of weight $\boldsymbol{\lambda}$ of the tensor product of k polynomial irreducible $\mathfrak{gl}_N$-modules with highest weights $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}$, respectively. We prove that the image of $\mathcal{B}_{\boldsymbol{K}}$ in $\mathrm{End}\left(\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}\right)$ is isomorphic to the algebra of functions on the intersection of suitable Schubert cycles in the Grassmannian of N-dimensional spaces of quasi-exponentials with exponents $\boldsymbol{K}$. We prove that the $\mathcal{B}_{\boldsymbol{K}}$-module $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$ is isomorphic to the coregular representation of that algebra of functions. We present a Bethe ansatz construction identifying the eigenvectors of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ with points of the considered intersection of Schubert cycles (cf [MTV7]).

The obtained result means that the eigenvectors of $\mathcal{B}_{\boldsymbol{K}}$ are in a bijective correspondence with suitable Nth-order differential operators with quasi-exponential kernel and prescribed singularities. This correspondence reduces the multidimensional problem of diagonalization of the Bethe algebra action to the one-dimensional problem of finding the corresponding differential operators. A separation of variables in a quantum integrable model is a reduction of a multidimensional spectral problem to a suitable one-dimensional problem, see for example Sklyanin's separation of variables in the $\mathfrak{gl}_2$ Gaudin model. In that respect, our correspondence can be viewed as 'a separation of variables' in the $\mathfrak{gl}_N$ Gaudin model associated with $\mathcal{B}_{\boldsymbol{K}}$ (cf [MTV7]).

The results of this paper and of [MTV6] are in the spirit of the $\mathfrak{gl}_N$ geometric Langlands correspondence, which, in particular, relates suitable commutative algebras of linear operators acting on $\mathfrak{gl}_N$-modules with the properties of schemes of suitable Nth-order differential operators.

The paper is organized as follows. In section 2, we discuss the representations of the current algebra $\mathfrak{gl}_N[t]$, in particular, Weyl modules. We introduce the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ as a subalgebra of $U(\mathfrak{gl}_N[t])$ in section 3. In section 4, we describe the affine space $\Omega_{\boldsymbol{\lambda}}$ of collections of N quasi-exponentials and discuss the properties of the algebra of functions on that space. In section 5, we introduce a collection of (Schubert) subvarieties in the space $\Omega_{\boldsymbol{\lambda}}$ and consider the algebra of functions on the intersection of the subvarieties. We prove the main results of the paper, theorems 6.3, 6.7, 6.9 and 6.12, in section 6. Section 7 describes the applications.

In this paper, we are using the same strategy as we used in [MTV6] with suitable technical modifications. We approach the final result on the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on a tensor product of irreducible polynomial $\mathfrak{gl}_N$-modules in three major steps. First, we consider the $\mathfrak{gl}_N[t]$-action on the space $\mathcal{V}^S$ of vector-valued polynomials in several variables $z_1, \ldots, z_n$, equivariant with respect to permutations of the variables (see section 2.5). The Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ acts on a weight subspace $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$. We show that the image of $\mathcal{B}_{\boldsymbol{K}}$ in $\mathrm{End}((\mathcal{V}^S)_{\boldsymbol{\lambda}})$ is a free polynomial algebra, and it can be naturally identified with the algebra of functions on the affine space $\Omega_{\boldsymbol{\lambda}}$ of collections of quasi-exponentials (see theorems 6.3 and 6.7).

The space $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$, the image of $\mathcal{B}_{\boldsymbol{K}}$ in $\mathrm{End}((\mathcal{V}^S)_{\boldsymbol{\lambda}})$ and the algebra of functions on $\Omega_{\boldsymbol{\lambda}}$ are modules over the algebra of symmetric polynomials in $z_1, \ldots, z_n$. For the second step we take the quotients by the ideal $I_{\boldsymbol{a}}$, given in section 2.4. As a result, we obtain an isomorphism of the image of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ acting on a weight subspace of a Weyl module over $\mathfrak{gl}_N[t]$ and the algebra of functions on a preimage of a point under the Wronski map $\pi : \Omega_{\boldsymbol{\lambda}} \to \mathbb{C}^n$, defined in section 4.4. Theorem 6.9 gives the precise statement.

The third step is to identify a weight subspace of a tensor product of irreducible polynomial $\mathfrak{gl}_N$-modules as a $\mathcal{B}_{\boldsymbol{K}}$-module with a suitable $\mathcal{B}_{\boldsymbol{K}}$-submodule of the corresponding weight subspace of the Weyl module. To get theorem 6.12, we show that such an identification amounts to a reduction of the algebra of functions on a preimage of a point under the Wronski map to the algebra of functions on the intersection of the Schubert subvarieties, introduced in section 5. Equality of dimensions (5.4), provided by Schubert calculus, plays a key role at the third step.

A reader familiar with the Bethe ansatz for the $\mathfrak{gl}_N$ Gaudin model can have an overview of the main results without proofs in sections 2.1, 2.2, 3.1, 7.1 and 8. This extraction can be read independently from the other content of the paper.

The results of this paper are related to the results on the $\mathfrak{gl}_N$-opers with an irregular singularity in the recent paper [FFR].

2. Representations of current algebra $\mathfrak{gl}_N[t]$

2.1. Lie algebra $\mathfrak{gl}_N$

Let $e_{ij}, i, j = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js}e_{ik} - \delta_{ik}e_{sj}$. We identify the Lie algebra $\mathfrak{sl}_N$ with the subalgebra in $\mathfrak{gl}_N$ generated by the elements $e_{ii} - e_{jj}$ and e_{ij} for $i \neq j, i, j = 1, \dots, N$. We denote by $\mathfrak{h} \subset \mathfrak{gl}_N$ the subalgebra generated by $e_{ii}, i = 1, \dots, N$. The subalgebra $\mathfrak{z}_N \subset \mathfrak{gl}_N$ generated by the element $\sum_{i=1}^N e_{ii}$ is central. The Lie algebra $\mathfrak{gl}_N$ is canonically isomorphic to the direct sum $\mathfrak{sl}_N \oplus \mathfrak{z}_N$.

Given an $N \times N$ matrix A with possibly noncommuting entries a_{ij}, we define its *row determinant* to be

$$\operatorname{rdet} A = \sum_{\sigma \in S_N} (-1)^\sigma a_{1\sigma(1)}a_{2\sigma(2)} \dots a_{N\sigma(N)}. \tag{2.1}$$

Let $Z(x)$ be the following polynomial in a variable x with coefficients in $U(\mathfrak{gl}_N)$:

$$Z(x) = \operatorname{rdet} \begin{pmatrix} x - e_{11} & -e_{21} & \dots & -e_{N1} \\ -e_{12} & x + 1 - e_{22} & \dots & -e_{N2} \\ \dots & \dots & \dots & \dots \\ -e_{1N} & -e_{2N} & \dots & x + N - 1 - e_{NN} \end{pmatrix}. \tag{2.2}$$

The next statement was proved in [HU] (see also [MNO, section 2.11]).

Theorem 2.1. *The coefficients of the polynomial $Z(x) - x^N$ are free generators of the center of $U(\mathfrak{gl}_N)$.*

Let M be a $\mathfrak{gl}_N$-module. A vector $v \in M$ has weight $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{ii}v = \lambda_i v$ for $i = 1, \dots, N$. A vector v is called *singular* if $e_{ij}v = 0$ for $1 \leqslant i < j \leqslant N$. If v is a singular of weight $\boldsymbol{\lambda}$, then

$$Z(x)v = \prod_{i=1}^N (x - \lambda_i + i - 1) \cdot v. \tag{2.3}$$

We denote by $(M)_{\boldsymbol{\lambda}}$ the subspace of M of weight $\boldsymbol{\lambda}$, by $(M)^{\mathrm{sing}}$ the subspace of M of all singular vectors and by $(M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ the subspace of M of all singular vectors of weight $\boldsymbol{\lambda}$.

Denote by $L_{\boldsymbol{\lambda}}$ the irreducible finite-dimensional $\mathfrak{gl}_N$-module with highest weight $\boldsymbol{\lambda}$. Any finite-dimensional $\mathfrak{gl}_N$-module M is isomorphic to the direct sum $\bigoplus_{\boldsymbol{\lambda}} L_{\boldsymbol{\lambda}} \otimes (M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$, where the spaces $(M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ are considered as trivial $\mathfrak{gl}_N$-modules.

The $\mathfrak{gl}_N$-module $L_{(1,0,\dots,0)}$ is the standard N-dimensional vector representation of $\mathfrak{gl}_N$. We denote it by V. We choose a highest weight vector in V and denote it by v_+.

A $\mathfrak{gl}_N$-module M is called polynomial if it is isomorphic to a submodule of $V^{\otimes n}$ for some n.

A sequence of integers $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0$ is called a *partition with at most N parts*. Set $|\boldsymbol{\lambda}| = \sum_{i=1}^{N} \lambda_i$. Then it is said that $\boldsymbol{\lambda}$ is a partition of $|\boldsymbol{\lambda}|$.

The $\mathfrak{gl}_N$-module $V^{\otimes n}$ contains the module $L_{\boldsymbol{\lambda}}$ if and only if $\boldsymbol{\lambda}$ is a partition of n with at most N parts.

For a Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

2.2. *Current algebra* $\mathfrak{gl}_N[t]$

Let $\mathfrak{gl}_N[t] = \mathfrak{gl}_N \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{gl}_N$-valued polynomials with the pointwise commutator. We call it the *current algebra*. We identify the Lie algebra $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$. Hence, any $\mathfrak{gl}_N[t]$-module has the canonical structure of a $\mathfrak{gl}_N$-module.

The standard generators of $\mathfrak{gl}_N[t]$ are $e_{ij} \otimes t^r$, $i, j = 1, \ldots, N$, $r \in \mathbb{Z}_{\geqslant 0}$. They satisfy the relations $[e_{ij} \otimes t^r, e_{sk} \otimes t^p] = \delta_{js} e_{ik} \otimes t^{r+p} - \delta_{ik} e_{sj} \otimes t^{r+p}$.

The subalgebra $\mathfrak{z}_N[t] \subset \mathfrak{gl}_N[t]$ generated by the elements $\sum_{i=1}^{N} e_{ii} \otimes t^r$, $r \in \mathbb{Z}_{\geqslant 0}$, is central. The Lie algebra $\mathfrak{gl}_N[t]$ is canonically isomorphic to the direct sum $\mathfrak{sl}_N[t] \oplus \mathfrak{z}_N[t]$.

It is convenient to collect elements of $\mathfrak{gl}_N[t]$ in generating series of a variable u. For $g \in \mathfrak{gl}_N$, set

$$g(u) = \sum_{s=0}^{\infty} (g \otimes t^s) u^{-s-1}.$$

For each $a \in \mathbb{C}$, there exists an automorphism ρ_a of $\mathfrak{gl}_N[t]$, $\rho_a : g(u) \mapsto g(u-a)$. Given a $\mathfrak{gl}_N[t]$-module M, we denote by $M(a)$ the pull-back of M through the automorphism ρ_a. As $\mathfrak{gl}_N$-modules, M and $M(a)$ are isomorphic by the identity map.

For any $\mathfrak{gl}_N[t]$-modules L, M and any $a \in \mathbb{C}$, the identity map $(L \otimes M)(a) \to L(a) \otimes M(a)$ is an isomorphism of $\mathfrak{gl}_N[t]$-modules.

We have the evaluation homomorphism, $ev : \mathfrak{gl}_N[t] \to \mathfrak{gl}_N$, $ev : g(u) \mapsto g u^{-1}$. Its restriction to the subalgebra $\mathfrak{gl}_N \subset \mathfrak{gl}_N[t]$ is the identity map. For any $\mathfrak{gl}_N$-module M, we denote by the same letter the $\mathfrak{gl}_N[t]$-module, obtained by pulling M back through the evaluation homomorphism. For each $a \in \mathbb{C}$, the $\mathfrak{gl}_N[t]$-module $M(a)$ is called an *evaluation module*.

If $b_1, \ldots, b_n$ are distinct complex numbers and $L_1, \ldots, L_n$ are irreducible finite-dimensional $\mathfrak{gl}_N$-modules, then the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^{n} L_s(b_s)$ is irreducible.

We have a natural $\mathbb{Z}_{\geqslant 0}$-grading on $\mathfrak{gl}_N[t]$ such that for any $g \in \mathfrak{gl}_N$, the degree of $g \otimes t^r$ equals r. We set the degree of u to be 1. Then the series $g(u)$ is homogeneous of degree -1.

A $\mathfrak{gl}_N[t]$-module is called *graded* if it has a $\mathbb{Z}_{\geqslant 0}$-grading compatible with the grading of $\mathfrak{gl}_N[t]$. Any irreducible graded $\mathfrak{gl}_N[t]$-module is isomorphic to an evaluation module $L(0)$ for some irreducible $\mathfrak{gl}_N$-module L (see [CG]).

Let M be a $\mathbb{Z}_{\geqslant 0}$-graded space with finite-dimensional homogeneous components. Let $M_j \subset M$ be the homogeneous component of degree j. We call the formal power series in a variable q,

$$\mathrm{ch}_M(q) = \sum_{j=0}^{\infty} (\dim M_j) q^j,$$

the *graded character* of M.

2.3. *Weyl modules*

Let W_m be the $\mathfrak{gl}_N[t]$-module generated by a vector v_m with the defining relations:

$$e_{ii}(u)v_m = \delta_{1i}\frac{m}{u}v_m, \qquad i = 1, \ldots, N,$$
$$e_{ij}(u)v_m = 0, \qquad 1 \leqslant i < j \leqslant N,$$
$$(e_{ji} \otimes 1)^{m\delta_{1i}+1}v_m = 0, \qquad 1 \leqslant i < j \leqslant N.$$

As an $\mathfrak{sl}_N[t]$-module, the module W_m is isomorphic to the Weyl module from [CL, CP], corresponding to the weight $m\omega_1$, where ω_1 is the first fundamental weight of $\mathfrak{sl}_N$. Note that $W_1 = V(0)$.

Lemma 2.2. *The module W_m has the following properties.*

(a) The module W_m has a unique grading such that W_m is a graded $\mathfrak{gl}_N[t]$-module and the degree of v_m equals 0.
(b) As a $\mathfrak{gl}_N$-module, W_m is isomorphic to $V^{\otimes m}$.
(c) A $\mathfrak{gl}_N[t]$-module M is an irreducible subquotient of W_m if and only if M has the form $L_{\boldsymbol{\lambda}}(0)$, where $\boldsymbol{\lambda}$ is a partition of m with at most N parts.

Proof. The first two properties are proved in [CP]. The third property follows from the first two. □

For each $b \in \mathbb{C}$, the $\mathfrak{gl}_N[t]$-module $W_m(b)$ is cyclic with a cyclic vector v_m.

Lemma 2.3. *The module $W_m(b)$ has the following properties.*

(i) As a $\mathfrak{gl}_N$-module, $W_m(b)$ is isomorphic to $V^{\otimes m}$.
(ii) A $\mathfrak{gl}_N[t]$-module M is an irreducible subquotient of $W_m(b)$ if and only if M has the form $L_{\boldsymbol{\lambda}}(b)$, where $\boldsymbol{\lambda}$ is a partition of m with N parts.
(iii) For any natural numbers $n_1, \ldots, n_k$ and distinct complex numbers $b_1, \ldots, b_k$, the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^k W_{n_s}(b_s)$ is cyclic with a cyclic vector $\otimes_{s=1}^k v_{n_s}$.
(iv) Let M be a cyclic finite-dimensional $\mathfrak{gl}_N[t]$-module with a cyclic vector v satisfying $e_{ij}(u)v = 0$ for $1 \leqslant i < j \leqslant N$, and $e_{ii}(u)v = \delta_{1i}\big(\sum_{s=1}^k n_s/(u-b_s)\big)v$ for $i = 1, \ldots, N$. Then there exists a surjective $\mathfrak{gl}_N[t]$-module homomorphism $\otimes_{s=1}^k W_{n_s}(b_s) \to M$ sending $\otimes_{s=1}^k v_{n_s}$ to v.

Given sequences $\boldsymbol{n} = (n_1, \ldots, n_k)$ of natural numbers and $\boldsymbol{b} = (b_1, \ldots, b_k)$ of distinct complex numbers, we call the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^k W_{n_s}(b_s)$ the *Weyl module associated with* $\boldsymbol{n}$ *and* $\boldsymbol{b}$.

Corollary 2.4. *A $\mathfrak{gl}_N[t]$-module M is an irreducible subquotient of $\otimes_{s=1}^k W_{n_s}(b_s)$ if and only if M has the form $\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)$, where $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}$ are partitions with at most N parts such that $|\boldsymbol{\lambda}^{(s)}| = n_s$, $s = 1, \ldots, k$.*

Consider the $\mathbb{Z}_{\geqslant 0}$-grading of the vector space W_m, introduced in lemma 2.2. Let W_m^j be the homogeneous component of W_m of degree j and $\bar{W}_m^j = \oplus_{r \geqslant j} W_m^r$. Since the $\mathfrak{gl}_N[t]$-module W_m is graded and $W_m = W_m(b)$ as vector spaces, $W_m(b) = \bar{W}_m^0 \supset \bar{W}_m^1 \supset \cdots$ is a descending filtration of $\mathfrak{gl}_N[t]$-submodules. This filtration induces the structure of the associated graded $\mathfrak{gl}_N[t]$-module on the vector space W_m which we denote by $\mathrm{gr}W_m(b)$.

Lemma 2.5. *The $\mathfrak{gl}_N[t]$-module $\mathrm{gr}W_m(b)$ is isomorphic to the evaluation module $(V^{\otimes m})(b)$.*

The space $\otimes_{s=1}^k W_{n_s}$ has a natural $\mathbb{Z}_{\geqslant 0}^k$-grading, induced by the gradings on the factors, and the associated descending $\mathbb{Z}_{\geqslant 0}^k$-filtration by the subspaces $\otimes_{s=1}^k \bar{W}_{n_s}^{j_s}$, invariant with respect to the $\mathfrak{gl}_N[t]$-action on the module $\otimes_{s=1}^k W_{n_s}(b_s)$. We denote by $\mathrm{gr}\big(\otimes_{s=1}^k W_{n_s}(b_s)\big)$ the induced structure of the associated graded $\mathfrak{gl}_N[t]$-module on the space $\otimes_{s=1}^k W_{n_s}$.

Lemma 2.6. *The $\mathfrak{gl}_N[t]$-modules $\mathrm{gr}\big(\otimes_{s=1}^k W_{n_s}(b_s)\big)$ and $\otimes_{s=1}^k \mathrm{gr} W_{n_s}(b_s)$ are canonically isomorphic.*

The proofs of lemmas 2.3, 2.5, 2.6, and corollary 2.4 can be found in [MTV6].

2.4. Remark on representations of symmetric group

Let S_n be the group of permutations of n elements. We denote by $\mathbb{C}[S_n]$ the regular representation of S_n. Given an S_n-module M we denote by M^S the subspace of all S_n-invariant vectors in M.

Lemma 2.7. *Let U be a finite-dimensional S_n-module. Then* $\dim(U \otimes \mathbb{C}[S_n])^S = \dim U$.

The group S_n acts on the algebra $\mathbb{C}[z_1, \dots, z_n]$ by permuting the variables. Let $\sigma_s(\boldsymbol{z}), s = 1, \dots, n$, be the sth elementary symmetric polynomial in $z_1, \dots, z_n$. The algebra of symmetric polynomials $\mathbb{C}[z_1, \dots, z_n]^S$ is a free polynomial algebra with generators $\sigma_1(\boldsymbol{z}), \dots, \sigma_n(\boldsymbol{z})$. It is well known that the algebra $\mathbb{C}[z_1, \dots, z_n]$ is a free $\mathbb{C}[z_1, \dots, z_n]^S$-module of rank $n!$ (see [M]).

Given $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, denote by $I_{\boldsymbol{a}} \subset \mathbb{C}[z_1, \dots, z_n]$ the ideal generated by the polynomials $\sigma_s(\boldsymbol{z}) - a_s, s = 1, \dots, n$. The ideal $I_{\boldsymbol{a}}$ is S_n-invariant.

Lemma 2.8. *For any $\boldsymbol{a} \in \mathbb{C}^n$, the S_n-representation $\mathbb{C}[z_1, \dots, z_n]/I_{\boldsymbol{a}}$ is isomorphic to the regular representation $\mathbb{C}[S_n]$.*

2.5. The $\mathfrak{gl}_N[t]$-module $\mathcal{V}^S$

Let $\mathcal{V}$ be the space of polynomials in $z_1, \dots, z_n$ with coefficients in $V^{\otimes n}$,

$$\mathcal{V} = V^{\otimes n} \otimes_{\mathbb{C}} \mathbb{C}[z_1, \dots, z_n].$$

The space $V^{\otimes n}$ is embedded in $\mathcal{V}$ as the subspace of constant polynomials.

Abusing notation, for any $v \in V^{\otimes n}$ and $p(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$, we will write $p(z_1, \dots, z_n)v$ instead of $v \otimes p(z_1, \dots, z_n)$.

We make the symmetric group S_n act on $\mathcal{V}$ by permuting the factors of $V^{\otimes n}$ and the variables $z_1, \dots, z_n$ simultaneously,

$$\sigma(p(z_1, \dots, z_n)v_1 \otimes \dots \otimes v_n) = p(z_{\sigma_1}, \dots, z_{\sigma_n})v_{(\sigma^{-1})_1} \otimes \dots \otimes v_{(\sigma^{-1})_n}, \qquad \sigma \in S_n.$$

We denote by $\mathcal{V}^S$ the subspace of S_n-invariants in $\mathcal{V}$.

Lemma 2.9 ([MTV6])**.** *The space $\mathcal{V}^S$ is a free $\mathbb{C}[z_1, \dots, z_n]^S$-module of rank N^n.*

We consider the space $\mathcal{V}$ as a $\mathfrak{gl}_N[t]$-module with the series $g(u)$, $g \in \mathfrak{gl}_N$, acting by

$$g(u)(p(z_1, \dots, z_n)v_1 \otimes \dots \otimes v_n) = p(z_1, \dots, z_n) \sum_{s=1}^n \frac{v_1 \otimes \dots \otimes g v_s \otimes \dots \otimes v_n}{u - z_s}. \tag{2.4}$$

Lemma 2.10 ([MTV6])**.** *The image of the subalgebra $U(\mathfrak{z}_N[t]) \subset U(\mathfrak{gl}_N[t])$ in* $\mathrm{End}(\mathcal{V})$ *coincides with the algebra of operators of multiplication by elements of $\mathbb{C}[z_1, \dots, z_n]^S$.*

The $\mathfrak{gl}_N[t]$-action on $\mathcal{V}$ commutes with the S_n-action. Hence, $\mathcal{V}^S$ is a $\mathfrak{gl}_N[t]$-submodule of $\mathcal{V}$.

Consider the grading on $\mathbb{C}[z_1, \ldots, z_n]$ such that $\deg z_i = 1$ for all $i = 1, \ldots, n$. We define a grading on $\mathcal{V}$ by setting $\deg(v \otimes p) = \deg p$ for any $v \in V^{\otimes n}$ and any $p \in \mathbb{C}[z_1, \ldots, z_n]$. The grading on $\mathcal{V}$ induces a natural grading on $\mathrm{End}(\mathcal{V})$.

Lemma 2.11. *The $\mathfrak{gl}_N[t]$-modules $\mathcal{V}$ and $\mathcal{V}^S$ are graded.*

The following lemma is contained in [K].

Lemma 2.12. *The $\mathfrak{gl}_N[t]$-module $\mathcal{V}^S$ is cyclic with a cyclic vector $v_+^{\otimes n}$.*

Lemma 2.13. *For any partition $\boldsymbol{\lambda}$ of n with at most N parts, the graded character of the space $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$ is given by*

$$\mathrm{ch}_{(\mathcal{V}^S)_{\boldsymbol{\lambda}}}(q) = \prod_{i=1}^{N} \frac{1}{(q)_{\lambda_i}}, \tag{2.5}$$

where $(q)_a = \prod_{j=1}^{a}(1 - q^j)$.

Proof. A basis of $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$ is given by the S_n-orbits of the $V^{\otimes n}$-valued polynomials of the form

$$p(z_1, \ldots, z_n)(v_+)^{\otimes \lambda_1} \otimes (e_{21}v_+)^{\otimes \lambda_2} \otimes \cdots \otimes (e_{N1}v_+)^{\otimes \lambda_N}$$

where $p(z_1, \ldots, z_n)$ is a polynomial symmetric with respect to the first λ_1 variables, symmetric with respect to the next λ_2 variables and so on, and finally symmetric with respect to the last λ_N variables. Clearly, the graded character of the space of such polynomials is given by formula (2.5). □

2.6. Weyl modules as quotients of $\mathcal{V}^S$

Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ be a sequence of complex numbers and $I_{\boldsymbol{a}} \subset \mathbb{C}[z_1, \ldots, z_n]$ the ideal, defined in section 2.4. Define

$$I_{\boldsymbol{a}}^{\mathcal{V}} = \left(V^{\otimes n} \bigoplus I_{\boldsymbol{a}}\right) \oplus \mathcal{V}^S. \tag{2.6}$$

Clearly, $I_{\boldsymbol{a}}^{\mathcal{V}}$ is a $\mathfrak{gl}_N[t]$-submodule of $\mathcal{V}^S$.

Introduce distinct complex numbers $b_1, \ldots, b_k$ and natural numbers $n_1, \ldots, n_k$ by the relation

$$\prod_{s=1}^{k}(u - b_s)^{n_s} = u^n + \sum_{j=1}^{n}(-1)^j a_j u^{n-j}. \tag{2.7}$$

Clearly, $\sum_{s=1}^{k} n_s = n$.

Lemma 2.14 ([MTV6]). *The $\mathfrak{gl}_N[t]$-modules $\mathcal{V}^S / I_{\boldsymbol{a}}^{\mathcal{V}}$ and $\otimes_{s=1}^{k} W_{n_s}(b_s)$ are isomorphic.*

3. Bethe algebra

3.1. Universal differential operator

Let $\boldsymbol{K} = (K_1, \ldots, K_N)$ be a sequence of distinct complex numbers. Let ∂ be the operator of differentiation in a variable u. Define the *universal differential operator* $\mathcal{D}^{\mathcal{B}}$ by

$$\mathcal{D}^{\mathcal{B}} = \mathrm{rdet}\begin{pmatrix} \partial - K_1 - e_{11}(u) & -e_{21}(u) & \ldots & -e_{N1}(u) \\ -e_{12}(u) & \partial - K_2 - e_{22}(u) & \ldots & -e_{N2}(u) \\ \ldots & \ldots & \ldots & \ldots \\ -e_{1N}(u) & -e_{2N}(u) & \ldots & \partial - K_N - e_{NN}(u) \end{pmatrix}.$$

It is a differential operator in the variable u, whose coefficients are formal power series in u^{-1} with coefficients in $U(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^{\mathcal{B}} = \partial^N + \sum_{i=1}^{N} B_i(u)\partial^{N-i}, \tag{3.1}$$

where

$$B_i(u) = \sum_{j=0}^{\infty} B_{ij}u^{-j} \tag{3.2}$$

and $B_{ij} \in U(\mathfrak{gl}_N[t])$ for $i = 1, \dots, N$, $j \geqslant 0$.

Lemma 3.1. *We have*

$$B_1(u) = -\sum_{i=1}^{N}(K_i + e_{ii}(u)) \tag{3.3}$$

and

$$\sum_{i=0}^{N} B_{i0}\alpha^{N-i} = \prod_{i=1}^{N}(\alpha - K_i), \tag{3.4}$$

where α *is a variable and* $B_{00} = 1$.

Lemma 3.2. *The element* $B_{ij} \in U(\mathfrak{gl}_N[t]) i = 1, \dots, N$, $j \geqslant 1$ *is a sum of homogeneous elements of degrees* $j-1, j-2, \dots, \max(j-i, 0)$.

Proof. It is straightforward to see that the series $B_i(u) - \sigma_i(K_1, \dots, K_N)$, where σ_i is the ith elementary symmetric polynomial, is a sum of homogeneous series of degrees $-1, \dots, -i$. The lemma follows. □

We call the unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by B_{ij}, with $i = 1, \dots, N$, $j \geqslant 0$, the *Bethe algebra* and denote it by $\mathcal{B}$.

Theorem 3.3 ([CT, MTV1])**.** *The algebra* $\mathcal{B}$ *is commutative. The algebra* $\mathcal{B}$ *commutes with the subalgebra* $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N[t])$.

3.2. $\mathcal{B}$*-modules*

Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$ be a sequence of partitions with at most N parts and $b_1, \dots, b_k$ distinct complex numbers.

Let $A_1(u), \dots, A_N(u)$ be the Laurent series in u^{-1} obtained by projecting coefficients of the series $B_i(u)\prod_{s=1}^{k}(u-b_s)^i$ to $\mathrm{End}\big(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\big)$.

The following lemma was proved in [MTV2].

Lemma 3.4. *The series* $A_1(u), \dots, A_N(u)$ *are polynomials in u. Moreover, the operators* $A_i(b_s)$, $i = 1, \dots, N$, $s = 1, \dots, k$, *are proportional to the identity operator, and*

$$\sum_{i=0}^{N} A_i(b_s) \prod_{j=0}^{N-i-1} (\alpha - j) = \prod_{l=1}^{N} \big(\alpha - \lambda_l^{(s)} - N + l\big), \qquad s = 1, \dots, k,$$

where α *is a variable and* $A_0(u) = 1$.

Proof. Consider the homomorphism $U(\mathfrak{gl}_N[t]) \to (U(\mathfrak{gl}_N))^{\otimes k}$,

$$g(u) \mapsto \sum_{s=1}^{k} \frac{1^{\otimes(s-1)} \otimes g \otimes 1^{\otimes(k-s)}}{u - b_s}, \qquad g \in \mathfrak{gl}_N. \tag{3.5}$$

Let $\tilde{A}_i(u), i = 1, \ldots, N$, be the Laurent series in u^{-1} obtained by projecting coefficients of the series $B_i(u)\prod_{s=1}^{k}(u - b_s)^i$ to $(U(\mathfrak{gl}_N))^{\otimes k}$. The series $\tilde{A}_1(u), \ldots, \tilde{A}_N(u)$ are polynomials in u, and by a straightforward calculation

$$\sum_{i=0}^{N} \tilde{A}_i(b_s) \prod_{j=0}^{N-i-1} (\alpha - j) = 1^{\otimes(s-1)} \otimes Z(\alpha - N + 1) \otimes 1^{\otimes(k-s)}, \qquad s = 1, \ldots, k,$$

where $\tilde{A}_0(u) = 1$. The lemma follows from theorem 2.1 and formula (2.3). □

Let $C_i(u), i = 1, \ldots, N$, be the Laurent series in u^{-1} obtained by projecting coefficients of the series $B_i(u)\prod_{s=1}^{n}(u - z_s)$ to $\mathrm{End}(\mathcal{V}^S)$.

Lemma 3.5. *The series $C_1(u), \ldots, C_N(u)$ are polynomials in u.*

Proof. The statement is a corollary of theorem 2.1 in [MTV3]. □

Set $n_s = |\boldsymbol{\lambda}^{(s)}|, s = 1, \ldots, k$. Let $\bar{C}_i(u), i = 1, \ldots, N$, be the Laurent series in u^{-1} obtained by projecting coefficients of the series $B_i(u)\prod_{s=1}^{k}(u - b_s)^{n_s}$ to $\mathrm{End}\big(\otimes_{s=1}^{k} W_{n_s}(b_s)\big)$.

Corollary 3.6. *The series $\bar{C}_1(u), \ldots, \bar{C}_N(u)$ are polynomials in u.*

Proof. The claim follows from lemmas 2.14 and 3.5. □

Corollary 3.7. *The products $A_i(u)\prod_{s=1}^{k}(u - b_s)^{n_s - i}, i = 1, \ldots, k$, are polynomials in u.*

Proof. The claim follows from lemma 2.4 and corollary 3.6. □

Let M be a $\mathfrak{gl}_N[t]$-module. As a subalgebra of $U(\mathfrak{gl}_N[t])$, the algebra $\mathcal{B}$ acts on M. If $H \subset M$ is a $\mathcal{B}$-invariant subspace, then we call the image of $\mathcal{B}$ in $\mathrm{End}(H)$ the *Bethe algebra associated with* H. Since $\mathcal{B}$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces $(M)_{\boldsymbol{\lambda}}$.

In what follows we study the action of the Bethe algebra $\mathcal{B}$ on the following $\mathcal{B}$-modules: $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$, $\big(\otimes_{s=1}^{k} W_{n_s}(b_s)\big)_{\boldsymbol{\lambda}}$, $\big(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\big)_{\boldsymbol{\lambda}}$.

4. Spaces of quasi-exponentials and the Wronski map

4.1. Spaces of quasi-exponentials

Let $\boldsymbol{K} = (K_1, \ldots, K_N)$ be a sequence of distinct complex numbers. Let $\boldsymbol{\lambda}$ be a partition of n with at most N parts. Let $\Omega_{\boldsymbol{\lambda}}$ be the affine n-dimensional space with coordinates $f_{ij}, i = 1, \ldots, N, j = 1, \ldots, \lambda_i$.

Introduce

$$f_i(u) = \mathrm{e}^{K_i u}(u^{\lambda_i} + f_{i1}u^{\lambda_i - 1} + \cdots + f_{i\lambda_i}), \qquad i = 1, \ldots, N. \tag{4.1}$$

We identify points $X \in \Omega_{\boldsymbol{\lambda}}$ with N-dimensional complex vector spaces generated by quasi-exponentials

$$f_i(u, X) = \mathrm{e}^{K_i u}(u^{\lambda_i} + f_{i1}(X)u^{\lambda_i - 1} + \cdots + f_{i\lambda_i}(X)), \qquad i = 1, \ldots, N. \tag{4.2}$$

Denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the algebra of regular functions on $\Omega_{\boldsymbol{\lambda}}$. It is the polynomial algebra in the variables f_{ij}. Define a grading on $\mathcal{O}_{\boldsymbol{\lambda}}$ such that the degree of the generator f_{ij} equals j for all (i, j).

Lemma 4.1. *The graded character of $\mathcal{O}_{\boldsymbol{\lambda}}$ is given by the formula*

$$\mathrm{ch}_{\mathcal{O}_{\boldsymbol{\lambda}}}(q) = \prod_{i=1}^{N} \frac{1}{(q)_{\lambda_i}}.$$

4.2. Another realization of $\mathcal{O}_{\boldsymbol{\lambda}}$

For arbitrary functions $g_1(u), \dots, g_N(u)$, introduce the Wronskian by the formula

$$\mathrm{Wr}(g_1(u), \dots, g_N(u)) = \det \begin{pmatrix} g_1(u) & g_1'(u) & \dots & g_1^{(N-1)}(u) \\ g_2(u) & g_2'(u) & \dots & g_2^{(N-1)}(u) \\ \dots & \dots & \dots & \dots \\ g_N(u) & g_N'(u) & \dots & g_N^{(N-1)}(u) \end{pmatrix}.$$

Let $f_i(u)$, $i = 1, \dots, N$, be the generating functions given by (4.1). We have

$$\mathrm{Wr}(f_1(u), \dots, f_N(u)) = \mathrm{e}^{\sum_{i=1}^{N} K_i u} \prod_{1\leqslant i<j\leqslant N} (K_j - K_i) \left(u^n + \sum_{s=1}^{n} (-1)^s \Sigma_s u^{n-s} \right), \tag{4.3}$$

where $\Sigma_1, \dots, \Sigma_n$ are elements of $\mathcal{O}_{\boldsymbol{\lambda}}$. Define the differential operator $\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}}$ by

$$\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} = \frac{1}{\mathrm{Wr}(f_1(u), \dots, f_N(u))} \mathrm{rdet} \begin{pmatrix} f_1(u) & f_1'(u) & \dots & f_1^{(N)}(u) \\ f_2(u) & f_2'(u) & \dots & f_2^{(N)}(u) \\ \dots & \dots & \dots & \dots \\ 1 & \partial & \dots & \partial^N \end{pmatrix}. \tag{4.4}$$

It is a differential operator in the variable u, whose coefficients are formal power series in u^{-1} with coefficients in $\mathcal{O}_{\boldsymbol{\lambda}}$,

$$\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} = \partial^N + \sum_{i=1}^{N} F_i(u) \partial^{N-i}, \tag{4.5}$$

where

$$F_i(u) = \sum_{j=0}^{\infty} F_{ij} u^{-j}, \tag{4.6}$$

and $F_{ij} \in \mathcal{O}_{\boldsymbol{\lambda}}$, $i = 1, \dots, N$, $j \geqslant 0$.

Define the *characteristic polynomial of the operator $\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}}$ at infinity* by

$$\chi(\alpha) = \sum_{i=0}^{N} F_{i0} \alpha^{N-i}, \tag{4.7}$$

where α is a variable and $F_{00} = 1$.

Lemma 4.2. *We have*

$$\chi(\alpha) = \prod_{i=1}^{N} (\alpha - K_i), \qquad \sum_{i=1}^{N} F_{i1} \alpha^{N-i} = \sum_{i=1}^{N} \lambda_i \prod_{\substack{j=1 \\ j\neq i}}^{N} (\alpha - K_j).$$

Proof. We have $\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} f_i(u) = 0$ for all $i = 1, \ldots, N$. Taking the coefficient of u^{λ_i} of the series $\mathrm{e}^{-K_i u}\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} f_i(u)$, we get $\chi(K_i) = 0$, for all $i = 1, \ldots, N$. This implies the first equality. The second equality follows similarly from considering the coefficient of $u^{\lambda_i - 1}$ of the series $\mathrm{e}^{-K_i u}\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} f_i(u)$. □

Lemma 4.3. *The functions $F_{ij} \in \mathcal{O}_{\boldsymbol{\lambda}}$, $i = 1, \ldots, N$, $j \geqslant 0$, generate the algebra $\mathcal{O}_{\boldsymbol{\lambda}}$.*

Proof. The coefficient of $u^{\lambda_i - j - 1}$ of the series $\mathrm{e}^{-K_i u}\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} f_i(u)$ has the form

$$-j f_{ij} \prod_{\substack{j=1\\ j\neq i}}^{N} (K_i - K_j) + \sum_{l=1}^{N} \left(\sum_{r=0}^{1} \sum_{s=0}^{j-1} c_{ijlrs} F_{lr} f_{is} + \sum_{r=2}^{j} \sum_{s=0}^{j-r+1} c_{ijlrs} F_{lr} f_{is} \right), \tag{4.8}$$

where c_{ijlrs} are some numbers. Since $\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}} f_i(u) = 0$, we can express recursively the elements f_{ij} via the elements F_{lr} starting with $j = 1$ and then increasing the second index j. □

4.3. Frobenius algebras

In this section, we recall some simple facts from commutative algebra. The word *algebra* will stand for an associative unital algebra over $\mathbb{C}$.

Let A be a commutative algebra. The algebra A considered as an A-module is called the *regular representation* of A. The dual space A^* is naturally an A-module, which is called the *coregular representation*.

Clearly, the image of A in $\mathrm{End}(A)$ for the regular representation is a maximal commutative subalgebra. If A is finite dimensional, then the image of A in $\mathrm{End}(A^*)$ for the coregular representation is a maximal commutative subalgebra as well.

If M is an A-module and $v \in M$ is an eigenvector of the A-action on M with eigenvalue $\xi_v \in A^*$, that is, $av = \xi_v(a)v$ for any $a \in A$, then ξ_v is a character of A, that is, $\xi_v(ab) = \xi_v(a)\xi_v(b)$.

If an element $v \in A^*$ is an eigenvector of the coregular action of A, then v is proportional to the character ξ_v. Moreover, each character $\xi \in A^*$ is an eigenvector of the coregular action of A and the corresponding eigenvalue equals ξ.

A nonzero element $\xi \in A^*$ is proportional to a character if and only if $\ker\xi \subset A$ is an ideal. Clearly, $A/\ker\xi \simeq \mathbb{C}$. On the other hand, if $\mathfrak{m} \subset A$ is an ideal such that $A/\mathfrak{m} \simeq \mathbb{C}$, then $\mathfrak{m}$ is a maximal proper ideal and $\mathfrak{m} = \ker\zeta$ for some character ζ.

A commutative algebra A is called *local* if it has a unique ideal $\mathfrak{m}$ such that $A/\mathfrak{m} \simeq \mathbb{C}$. In other words, a commutative algebra A is local if it has a unique character. It is easy to see that any proper ideal of the local algebra A is contained in the ideal $\mathfrak{m}$.

It is known that any finite-dimensional commutative algebra A is isomorphic to a direct sum of local algebras, and the local summands are in bijection with characters of A.

Let A be a commutative algebra. A bilinear form $(,) : A \otimes A \to \mathbb{C}$ is called *invariant* if $(ab, c) = (a, bc)$ for all $a, b, c \in A$.

A finite-dimensional commutative algebra A which admits an invariant nondegenerate symmetric bilinear form $(,) : A \otimes A \to \mathbb{C}$ is called a *Frobenius algebra*. It is easy to see that distinct local summands of a Frobenius algebra are orthogonal.

The following properties of Frobenius algebras will be useful.

Lemma 4.4. *A finite direct sum of Frobenius algebras is a Frobenius algebra.*

Let A be a Frobenius algebra. Let $I \subset A$ be a subspace. Denote by $I^{\perp} \subset A$ the orthogonal complement to I. Then $\dim I + \dim I^{\perp} = \dim A$, and the subspace I is an ideal if and only if $I^{\perp}$ is an ideal.

Let A_0 be a local Frobenius algebra with maximal ideal $\mathfrak{m} \subset A_0$. Then $\mathfrak{m}^\perp$ is a one-dimensional ideal. Let $m^\perp \in \mathfrak{m}^\perp$ be an element such that $(1, m^\perp) = 1$.

Lemma 4.5. *Any nonzero ideal* $I \subset A_0$ *contains* $\mathfrak{m}^\perp$.

For a subset $I \subset A$ define its annihilator as $\operatorname{Ann} I = \{a \in A, |aI = 0\}$. The annihilator $\operatorname{Ann} I$ is an ideal.

Lemma 4.6. *Let A be a Frobenius algebra and $I \subset A$ an ideal. Then* $\operatorname{Ann} I = I^\perp$. *In particular,* $\dim I + \dim \operatorname{Ann} I = \dim A$.

For any ideal $I \subset A$, the regular action of A on itself induces an action of A/I on $\operatorname{Ann} I$.

Lemma 4.7. *The A/I-module* $\operatorname{Ann} I$ *is isomorphic to the coregular representation of A/I. In particular, the image of A/I in* $\operatorname{End}(\operatorname{Ann} I)$ *is a maximal commutative subalgebra.*

Let $P_1, \ldots, P_m$ be polynomials in variables $x_1, \ldots, x_m$. Denote by I the ideal in $\mathbb{C}[x_1, \ldots, x_m]$ generated by $P_1, \ldots, P_m$.

Lemma 4.8. *If the algebra $\mathbb{C}[x_1, \ldots, x_m]/I$ is nonzero and finite dimensional, then it is a Frobenius algebra.*

The proofs of lemmas 4.4–4.8 can be found in [MTV6].

Lemma 4.8 has the following generalization. Let $\mathbb{C}_T(x_1, \ldots, x_m)$ be the algebra of rational functions in $x_1, \ldots, x_m$, regular at points of a subset $T \subset \mathbb{C}^m$. Denote by I_T the ideal in $\mathbb{C}_T(x_1, \ldots, x_m)$ generated by $P_1, \ldots, P_m$.

Lemma 4.9. *Assume that the solution set to the system of equations*

$$P_1(x_1, \ldots, x_m) = \cdots = P_m(x_1, \ldots, x_m) = 0$$

is finite and lies in T. Then the algebra $\mathbb{C}_T(x_1, \ldots, x_m)/I_T$ is a Frobenius algebra.

4.4. Wronski map

Let X be a point of $\Omega_{\boldsymbol{\lambda}}$. Define

$$\mathrm{Wr}_X(u) = \mathrm{Wr}(f_1(u, X), \ldots, f_N(u, X)), \tag{4.9}$$

where $f_1(u, X), \ldots, f_N(u, X)$ are given by (4.2). Define the *Wronski map* $\pi : \Omega_{\boldsymbol{\lambda}} \to \mathbb{C}^n$ by $X \mapsto \boldsymbol{a} = (a_1, \ldots, a_n)$ if

$$\mathrm{Wr}_X(u) = \mathrm{e}^{\sum_{i=1}^N K_i u} \prod_{1\leqslant i<j\leqslant N} (K_j - K_i) \left(u^n + \sum_{s=1}^n (-1)^s a_s u^{n-s} \right).$$

For $\boldsymbol{a} \in \mathbb{C}^n$, let $I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}}$ be the ideal in $\mathcal{O}_{\boldsymbol{\lambda}}$ generated by the elements $\Sigma_s - a_s$, $s = 1, \ldots, n$, where $\Sigma_1, \ldots, \Sigma_n$ are defined by (4.3). The quotient algebra

$$\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} = \mathcal{O}_{\boldsymbol{\lambda}} \big/ I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}} \tag{4.10}$$

is the scheme-theoretic fiber of the Wronski map. We call it the *algebra of functions on the preimage* $\pi^{-1}(\boldsymbol{a})$.

Lemma 4.10.

(i) The algebra $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is a finite-dimensional commutative associative unital algebra and $\dim_{\mathbb{C}} \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ *does not depend on* $\boldsymbol{a}$.

(ii) The algebra $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is a Frobenius algebra.

Proof. The Wronski map is a polynomial map of finite degree (see propositions 4.2 and 3.1 in [MTV5]). This implies part (i) of the lemma and the fact that $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is a direct sum of local algebras. The dimension of $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is the degree of the Wronski map and the local summands correspond to the points of the set $\pi^{-1}(\boldsymbol{a})$. The algebra $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is Frobenius by lemma 4.8. □

5. Intersection $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ and algebra $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$

5.1. Intersection $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$

For $b \in \mathbb{C}$ and a partition $\boldsymbol{\mu}$ of n with at most N parts, denote by $\Omega_{\boldsymbol{\mu}}(b)$ the variety of all spaces of quasi-exponentials $X \in \Omega_{\boldsymbol{\lambda}}$ such that for every $i = 1, \ldots, N$ there exists a function $g(u) \in X$ with zero of order $\mu_i + N - i$ at b.

Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$ be a sequence of partitions with at most N parts such that $\sum_{s=1}^{k} |\boldsymbol{\lambda}^{(s)}| = n$. Denote $n_s = |\boldsymbol{\lambda}^{(s)}|$. Let $\boldsymbol{b} = (b_1, \ldots, b_k)$ be a sequence of distinct complex numbers.

Consider the intersection

$$\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \bigcap_{s=1}^{k} \Omega_{\boldsymbol{\lambda}^{(s)}}(b_s). \tag{5.1}$$

Given a space of quasi-exponentials $X \subset \Omega_{\boldsymbol{\lambda}}$, denote by $\mathcal{D}_X$ the monic scalar differential operator of order N with kernel X. The operator $\mathcal{D}_X$ equals the operator $\mathcal{D}^{\mathcal{O}}_{\boldsymbol{\lambda}}$, see (4.4), computed at X.

Lemma 5.1. *A space of quasi-exponentials $X \subset \Omega_{\boldsymbol{\lambda}}$ is a point of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ if and only if the singular points of the operator $\mathcal{D}_X$ are at $b_1, \ldots, b_k$ and ∞ only, the singular points at $b_1, \ldots, b_k$ are regular, and the exponents at $b_s, s = 1, \ldots, k$, are equal to $\lambda^{(s)}_N, \lambda^{(s)}_{N-1} + 1, \ldots, \lambda^{(s)}_1 + N - 1$.*

Lemma 5.2. *Let $\boldsymbol{b} = (b_1, \ldots, b_k)$, and $\boldsymbol{n} = (n_1, \ldots, n_k)$ be as at the beginning of this section. Let the numbers $\boldsymbol{a} = (a_1, \ldots, a_n)$ be related to $\boldsymbol{b}$ and $\boldsymbol{n}$ as in (2.7). Then $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \subset \pi^{-1}(\boldsymbol{a})$. In particular, the set $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is finite.*

Let $\mathcal{Q}_{\boldsymbol{\lambda}}$ be the field of fractions of $\mathcal{O}_{\boldsymbol{\lambda}}$, and $\mathcal{Q}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \subset \mathcal{Q}_{\boldsymbol{\lambda}}$ the subring of elements regular at all points of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.

Consider the $N \times N$ matrices $M_1, \ldots, M_k$ with entries in $\mathcal{O}_{\boldsymbol{\lambda}}$,

$$(M_s)_{ij} = \frac{1}{\left(\lambda^{(s)}_j + N - j\right)!} \left(\left(\frac{d}{du}\right)^{\lambda^{(s)}_j + N - j} f_i(u) \right)\Bigg|_{u=b_s}.$$

The values of $M_1, \ldots, M_k$ at any point of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{b}}$ are matrices invertible over $\mathbb{C}$. Therefore, the inverse matrices $M_1^{-1}, \ldots, M_k^{-1}$ exist as matrices with entries in $\mathcal{Q}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.

Introduce the elements $g_{ijs} \in \mathcal{Q}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, $i = 1, \ldots, N$, $j = 0, \ldots, d_1$, $s = 1, \ldots, k$, by the rule

$$\sum_{j=0}^{d_1} g_{ijs}(u - b_s)^j = \sum_{m=1}^{N} \left(M_s^{-1}\right)_{im} f_m(u). \tag{5.2}$$

Clearly, $g_{i,\lambda^{(s)}_j + N - j, s} = \delta_{ij}$ for all $i, j = 1, \ldots, N$, and $s = 1, \ldots, k$.

For each $s = 1, \ldots, k$, let $J^{\mathcal{Q},s}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ be the ideal in $\mathcal{Q}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ generated by the elements g_{ijs}, $i = 1, \ldots, N$, $j = 0, \ldots, \lambda^{(s)}_i + N - i - 1$, and $J^{\mathcal{Q}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \sum_{s=1}^{k} J^{\mathcal{Q},s}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. Note that the number of generators of the ideal $J^{\mathcal{Q}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ equals n.

The quotient algebra

$$\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} = \mathcal{Q}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \big/ J^{\mathcal{Q}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \tag{5.3}$$

is the scheme-theoretic intersection of varieties $\Omega_{\boldsymbol{\lambda}^{(s)}}$, $s = 1, \ldots, k$. We call it the *algebra of functions on* $\Omega_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.

Lemma 5.3. *The algebra* $\Omega_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *is a Frobenius algebra.*

Proof. The claim follows from lemma 4.9. □

It is known from Schubert calculus that

$$\dim \mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} = \dim \left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}, \tag{5.4}$$

see [MTV8, lemma 3.6 and proposition 3.7].

5.2. *Algebra* $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *as a quotient of* $\mathcal{O}_{\boldsymbol{\lambda}}$

Consider the differential operator

$$\tilde{\mathcal{D}}^{\mathcal{O}}_{\boldsymbol{\lambda}} = \operatorname{rdet} \begin{pmatrix} f_1(u) & f_1'(u) & \ldots & f_1^{(N)}(u) \\ f_2(u) & f_2'(u) & \ldots & f_2^{(N)}(u) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \partial & \ldots & \partial^N \end{pmatrix}. \tag{5.5}$$

It is a differential operator in the variable u whose coefficients are polynomials in u with coefficients in $\mathcal{O}_{\boldsymbol{\lambda}}$,

$$\tilde{\mathcal{D}}^{\mathcal{O}}_{\boldsymbol{\lambda}} = \sum_{i=0}^{N} G_i(u)\partial^{N-i}. \tag{5.6}$$

Clearly, $G_i(u) = 0$ for $i > n$, and $\deg G_i \leqslant n$, otherwise. We also have

$$G_0(u) = \operatorname{Wr}(f_1(u), \ldots, f_N(u)),$$
$$G_i(u) = \operatorname{Wr}(f_1(u), \ldots, f_N(u))F_i(u), \qquad i = 1, \ldots, N,$$

Introduce the elements $G_{ijs} \in \mathcal{O}_{\boldsymbol{\lambda}}$, $i = 0, \ldots, N$, $j = 0, \ldots, n - i$, $s = 1, \ldots, k$, by the rule

$$G_i(u) = \sum_{j=0}^{n} G_{ijs}(u - b_s)^j. \tag{5.7}$$

Define the *indicial polynomial* $\chi^{\mathcal{O}}_s(\alpha)$ *at* b_s by the formula

$$\chi^{\mathcal{O}}_s(\alpha) = \sum_{i=0}^{N} G_{i,n_s-i,s} \prod_{j=0}^{N-i-1} (\alpha - j).$$

It is a polynomial of degree N in the variable α with coefficients in $\mathcal{O}_{\boldsymbol{\lambda}}$.

Lemma 5.4. *For a complex number r, the element* $\chi^{\mathcal{O}}_s(r)$ *is invertible in* $\mathcal{Q}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *provided* $r \neq \lambda^{(s)}_j + N - j$ *for all* $j = 1, \ldots, N$.

Proof. An element of $\mathcal{Q}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ is invertible if and only if its value at any point of $\Omega_{\Lambda,\boldsymbol{b}}$ is nonzero. Now the claim follows from lemmas 5.1 and 5.2. □

For each $s = 1, \ldots, k$, let $I^{\mathcal{Q},s}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ be the ideal in $\mathcal{Q}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ generated by the elements $G_{ijs}, i = 0, \ldots, N, 0 \leqslant j < n_s - i$, and the coefficients of the polynomials

$$\chi^{\mathcal{O}}_s(\alpha) - \prod_{\substack{r=1\\ r\neq s}}^{k} (b_s - b_r)^{n_r} \prod_{l=1}^{N} \big(\alpha - \lambda^{(s)}_l - N + l\big), \qquad s = 1, \ldots, k. \tag{5.8}$$

Denote $I^{\mathcal{Q}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} = \sum_{s=1}^{k} I^{\mathcal{Q},s}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.

Lemma 5.5 ([MTV6]). *For any $s = 1, \ldots, k$, the ideals $I^{\mathcal{Q},s}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ and $J^{\mathcal{Q},s}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ coincide.*

Let $I^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ be the ideal in $\mathcal{O}_{\boldsymbol{\lambda}}$ generated by the elements $G_{ijs}, i = 0, \ldots, N, s = 1, \ldots, k, 0 \leqslant j < n_s - i$, and the coefficients of polynomials (5.8).

Proposition 5.6. *The algebra $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ is isomorphic to the quotient algebra $\mathcal{O}_{\boldsymbol{\lambda}}\big/I^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.*

Proof. By lemma 5.5, the ideals $I^{\mathcal{Q}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ and $J^{\mathcal{Q}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ coincide, so the algebra $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ is isomorphic to the quotient algebra $\mathcal{Q}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big/I^{\mathcal{Q}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$. By lemma 5.1, the algebraic set defined by the ideal $I^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ equals $\Omega_{\Lambda,\boldsymbol{b}}$. The set $\Omega_{\Lambda,\boldsymbol{b}}$ is finite by lemma 5.2. Therefore, the quotient algebras $\mathcal{Q}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big/I^{\mathcal{Q}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ and $\mathcal{O}_{\boldsymbol{\lambda}}\big/I^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ are isomorphic. □

5.3. Algebra $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ as a quotient of $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$

Recall that $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} = \mathcal{O}_{\boldsymbol{\lambda}}\big/I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is the algebra of functions on $\pi^{-1}(\boldsymbol{a})$ (see (4.10)). For an element $F \in \mathcal{O}_{\boldsymbol{\lambda}}$, we denote by $\bar{F}$ the projection of F to the quotient algebra $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$.

Define the *indicial polynomial* $\bar{\chi}^{\mathcal{O}}_s(\alpha)$ *at* b_s by the formula

$$\bar{\chi}^{\mathcal{O}}_s(\alpha) = \sum_{i=0}^{N} \bar{G}_{i,n_s-i,s} \prod_{j=0}^{N-i-1} (\alpha - j).$$

Let $\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ be the ideal in $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ generated by the elements $\bar{G}_{ijs}, i = 1, \ldots, N, s = 1, \ldots, k, 0 \leqslant j < n_s - i$, and the coefficients of the polynomials

$$\bar{\chi}^{\mathcal{O}}_s(\alpha) - \prod_{\substack{r=1\\ r\neq s}}^{k} (b_s - b_r)^{n_r} \prod_{l=1}^{N} \big(\alpha - \lambda^{(s)}_l - N + l\big), \qquad s = 1, \ldots, k.$$

Proposition 5.7. *The algebra $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ is isomorphic to the quotient algebra $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}\big/\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.*

Proof. It is easy to see that the elements $G_{0js}, j = 0, \ldots, n_s - 1, s = 1, \ldots, k$, generate the ideal $I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}}$ in $\mathcal{O}_{\boldsymbol{\lambda}}$. Moreover, the projection of the ideal $I^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \subset \mathcal{O}_{\boldsymbol{\lambda}}$ to $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ equals $\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$. Hence, the claim follows from proposition 5.6. □

Recall that the ideal $\mathrm{Ann}\big(\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big) \subset \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is naturally an $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module.

Corollary 5.8. *The $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module $\mathrm{Ann}\big(\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big)$ is isomorphic to the coregular representation of $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ on the dual space $(\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^*$.*

Proof. The statement follows from lemmas 5.3 and 4.7. □

6. Three isomorphisms

6.1. Auxiliary lemma

Let $\boldsymbol{\lambda}$ be a partition of n with at most N parts. Recall that given a space of quasi-exponentials $X \in \Omega_{\boldsymbol{\lambda}}$, we denote by $\mathcal{D}_X$ the monic scalar differential operator of order N with kernel X.

Let M be a $\mathfrak{gl}_N[t]$-module M and v an eigenvector of the Bethe algebra $\mathcal{B} \subset U(\mathfrak{gl}_N[t])$ acting on M. Then for any coefficient $B_i(u)$ of the universal differential operator $\mathcal{D}^{\mathcal{B}}$ we have $B_i(u)v = h_i(u)v$, where $h_i(u)$ is a scalar series. We call the scalar differential operator

$$\mathcal{D}_v^{\mathcal{B}} = \partial^N + \sum_{i=1}^{N} h_i(u)\partial^{N-i} \tag{6.1}$$

the differential *operator associated with* v.

We consider $\mathbb{C}^n$ with the symmetric group S_n action defined by permutation of coordinates.

Lemma 6.1. *There exist a Zariski open S_n-invariant subset Θ of $\mathbb{C}^n$ and a Zariski open subset Ξ of $\Omega_{\boldsymbol{\lambda}}$ with the following properties.*

(i) For any $(b_1, \ldots, b_n) \in \Theta$, there exists a basis of $\left(\otimes_{s=1}^n V(b_s)\right)_{\boldsymbol{\lambda}}$ such that every basis vector v is an eigenvector of the Bethe algebra and $\mathcal{D}_v^{\mathcal{B}} = \mathcal{D}_X$ for some $X \in \Xi$. Moreover, different basis vectors correspond to different points of Ξ.

(ii) For any $X \in \Xi$, if $b_1, \ldots, b_n$ are all roots of the Wronskian Wr_X, then $(b_1, \ldots, b_n) \in \Theta$, and there exists a unique up to proportionality vector $v \in \left(\otimes_{s=1}^n V(b_s)\right)_{\boldsymbol{\lambda}}$ such that v is an eigenvector of the Bethe algebra with $\mathcal{D}_v^{\mathcal{B}} = \mathcal{D}_X$.

Proof. The basis in part (i) is constructed by the Bethe ansatz method as in section 10 of [MTV4]. The equality $\mathcal{D}_v^{\mathcal{B}} = \mathcal{D}_X$ is proved in [MTV1]. The existence of an eigenvector v in part (ii) for generic $X \subset \Omega_{\bar{\boldsymbol{\lambda}}}$ is proved as in section 10 of [MTV4]. □

Corollary 6.2. *The degree of the Wronski map equals* $\dim(V^{\otimes n})_{\boldsymbol{\lambda}}$.

6.2. Isomorphism of algebras $\mathcal{O}_{\boldsymbol{\lambda}}$ and $\mathcal{B}_{\boldsymbol{\lambda}}$

Consider the $\mathcal{B}$-module $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$. Denote $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$ by $\mathcal{M}_{\boldsymbol{\lambda}}$ and the Bethe algebra associated with $(\mathcal{V}^S)_{\boldsymbol{\lambda}}$ by $\mathcal{B}_{\boldsymbol{\lambda}}$.

Consider the map

$$\tau_{\boldsymbol{\lambda}} : \mathcal{O}_{\boldsymbol{\lambda}} \to \mathcal{B}_{\boldsymbol{\lambda}}, \qquad F_{ij} \mapsto \hat{B}_{ij},$$

where the elements $F_{ij} \in \mathcal{O}_{\boldsymbol{\lambda}}$ are defined by (4.6) and $\hat{B}_{ij} \in \mathcal{B}_{\boldsymbol{\lambda}}$ are the images of the elements $B_{ij} \in \mathcal{B}$, defined by (3.2).

Theorem 6.3. *The map $\tau_{\boldsymbol{\lambda}}$ is a well-defined isomorphism of algebras.*

Proof. Let a polynomial $R(F_{ij})$ in generators F_{ij} be equal to zero in $\mathcal{O}_{\boldsymbol{\lambda}}$. Let us prove that the corresponding polynomial $R(\hat{B}_{ij})$ is equal to zero in the $\mathcal{B}_{\boldsymbol{\lambda}}$. Indeed, $R(\hat{B}_{ij})$ is a polynomial in $z_1, \ldots, z_n$ with values in $\mathrm{End}\big((V^{\otimes n})_{\boldsymbol{\lambda}}\big)$. Let Θ be the set, introduced in lemma 6.1, and $(b_1, \ldots, b_n) \in \Theta$. Then by part (i) of lemma 6.1, the value of the polynomial $R(\hat{B}_{ij})$ at $z_1 = b_1, \ldots, z_n = b_n$ equals zero. Hence, the polynomial $R(\hat{B}_{ij})$ equals zero identically and the map $\tau_{\boldsymbol{\lambda}}$ is well defined.

Let a polynomial $R(F_{ij})$ in generators F_{ij} be a nonzero element of $\mathcal{O}_{\boldsymbol{\lambda}}$. Then the value of $R(F_{ij})$ at a generic point $X \in \Omega_{\bar{\boldsymbol{\lambda}}}(\infty)$ is not equal to zero. Then by part (ii) of lemma 6.1, the polynomial $R(\hat{B}_{ij})$ is not identically equal to zero. Therefore, the map $\tau_{\boldsymbol{\lambda}}$ is injective.

Since the elements $\hat{B}_{ij}$ generate the algebra $\mathcal{B}_\lambda$, the map τ_λ is surjective. □

The algebra $\mathbb{C}[z_1, \ldots, z_n]^S$ is embedded into the algebra $\mathcal{B}_\lambda$ as the subalgebra of operators of multiplication by symmetric polynomials (see lemmas 2.10 and formula (3.4)). The algebra $\mathbb{C}[z_1, \ldots, z_n]^S$ is embedded into the algebra $\mathcal{O}_\lambda$, the elementary symmetric polynomials $\sigma_1(z), \ldots, \sigma_n(z)$ being mapped to the elements $\Sigma_1, \ldots, \Sigma_n$, defined by (4.3). These embeddings give the algebras $\mathcal{B}_\lambda$ and $\mathcal{O}_\lambda$ the structure of $\mathbb{C}[z_1, \ldots, z_n]^S$-modules.

Lemma 6.4. *The map $\tau_\lambda : \mathcal{O}_\lambda \to \mathcal{B}_\lambda$ is an isomorphism of $\mathbb{C}[z_1, \ldots, z_n]^S$-modules, that is, $\tau_\lambda(\Sigma_i) = \sigma_i(z)$ for all $i = 1, \ldots, n$.*

Proof. The claim follows from the fact that

$$F_1(u) = -\frac{\mathrm{Wr}'(f_1(u), \ldots, f_N(u))}{\mathrm{Wr}(f_1(u), \ldots, f_N(u))} \tag{6.2}$$

where $'$ denotes the derivative with respect to u, and from formula (3.3). □

Lemma 6.5. *For any homogeneous element $F \in \mathcal{O}_\lambda$, the degrees of homogeneous components of $\tau_\lambda(F) \in \mathcal{B}_\lambda$ do not exceed* $\deg F$.

Proof. It suffices to prove the claim for the generators $f_{ij} \in \mathcal{O}_\lambda$. In that case, the statement follows from formula (4.8) and lemma 3.2 by induction with respect to j, starting from $j = 1$. □

Given a vector $v \in \mathcal{M}_\lambda$, consider a linear map

$$\mu_v : \mathcal{O}_\lambda \to \mathcal{M}_\lambda, \qquad F \mapsto \tau_\lambda(F)v.$$

Lemma 6.6. *If $v \in \mathcal{M}_\lambda$ is nonzero, then the map μ_v is injective.*

Proof. The algebra $\mathcal{O}_\lambda$ is a free polynomial algebra containing the subalgebra $\mathbb{C}[z_1, \ldots, z_n]^S$. By part (i) of lemma 4.10, the quotient algebra $\mathcal{O}_\lambda/\mathbb{C}[z_1, \ldots, z_n]^S$ is finite dimensional. The kernel of μ_v is an ideal in $\mathcal{B}_\lambda$ which has zero intersection with $\mathbb{C}[z_1, \ldots, z_n]^S$ and, therefore, is the zero ideal. □

The graded character of $\mathcal{V}_\lambda^S$ is given by formula (2.5). Fix a nonzero vector $v \in \mathcal{V}_\lambda^S$ of degree 0. Such a vector is unique up to multiplication by a nonzero number. Then the map μ_v will be denoted by μ_λ.

Theorem 6.7. *The map $\mu_\lambda : \mathcal{O}_\lambda \to \mathcal{M}_\lambda$ is a vector isomorphism. This isomorphism preserves the degree of elements. The maps τ_λ and μ_λ intertwine the action of multiplication operators on $\mathcal{O}_\lambda$ and the action of the Bethe algebra $\mathcal{B}_\lambda$ on $\mathcal{M}_\lambda$, that is, for any $F, G \in \mathcal{O}_\lambda$, we have*

$$\mu_\lambda(FG) = \tau_\lambda(F)\mu_\lambda(G). \tag{6.3}$$

In other words, the maps τ_λ and μ_λ give an isomorphism of the regular representation of $\mathcal{O}_\lambda$ and the $\mathcal{B}_\lambda$-module $\mathcal{M}_\lambda$.

Proof. The map μ_λ is injective by lemma 6.6. The map μ_λ does not increase the degree by lemma 6.5. The graded characters of $\mathcal{O}_\lambda$ and $\mathcal{M}_\lambda$ are the same by lemmas 4.1 and 2.13. Hence, the map μ_λ is surjective. Formula (6.3) follows from theorem 6.3. □

6.3. Isomorphism of algebras $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$

Let $\boldsymbol{a} = (a_1, \ldots, a_N)$ be a sequence of complex numbers. Let distinct complex numbers $b_1, \ldots, b_k$ and integers $n_1, \ldots, n_k$ be given by (2.7).

Let $I^{\mathcal{B}}_{\boldsymbol{\lambda},\boldsymbol{a}} \subset \mathcal{B}_{\boldsymbol{\lambda}}$ be the ideal generated by the elements $\sigma_i(\boldsymbol{z}) - a_i$, $i = 1, \ldots, n$. Consider the subspace $I^{\mathcal{M}}_{\boldsymbol{\lambda},\boldsymbol{a}} = I^{\mathcal{B}}_{\boldsymbol{\lambda},\boldsymbol{a}}\mathcal{M}_{\boldsymbol{\lambda}}$, where $I^{\mathcal{V}}_{\boldsymbol{a}}$ is given by (2.6). Recall that the ideal $I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is defined in section 4.4.

Lemma 6.8. *We have*

$$\tau_{\boldsymbol{\lambda}}\big(I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}}\big) = I^{\mathcal{B}}_{\boldsymbol{\lambda},\boldsymbol{a}}, \qquad \mu_{\boldsymbol{\lambda}}\big(I^{\mathcal{O}}_{\boldsymbol{\lambda},\boldsymbol{a}}\big) = I^{\mathcal{M}}_{\boldsymbol{\lambda},\boldsymbol{a}}, \qquad \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} = \mathcal{B}_{\boldsymbol{\lambda}}\big/I^{\mathcal{B}}_{\boldsymbol{\lambda},\boldsymbol{a}}, \qquad \mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}} = \mathcal{M}_{\boldsymbol{\lambda}}\big/I^{\mathcal{M}}_{\boldsymbol{\lambda},\boldsymbol{a}}.$$

Proof. The lemma follows from theorems 6.3, 6.7 and lemmas 6.4, 2.14. □

By lemma 6.8, the maps $\tau_{\boldsymbol{\lambda}}$ and $\mu_{\boldsymbol{\lambda}}$ induce the maps

$$\tau_{\boldsymbol{\lambda},\boldsymbol{a}} : \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}, \qquad \mu_{\boldsymbol{\lambda},\boldsymbol{a}} : \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}. \tag{6.4}$$

Theorem 6.9. *The map $\tau_{\boldsymbol{\lambda},\boldsymbol{a}}$ is an isomorphism of algebras. The map $\mu_{\boldsymbol{\lambda},\boldsymbol{a}}$ is an isomorphism of vector spaces. The maps $\tau_{\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mu_{\boldsymbol{\lambda},\boldsymbol{a}}$ intertwine the action of multiplication operators on $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ and the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$ on $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$, that is, for any $F, G \in \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$, we have*

$$\mu_{\boldsymbol{\lambda},\boldsymbol{a}}(FG) = \tau_{\boldsymbol{\lambda},\boldsymbol{a}}(F)\mu_{\boldsymbol{\lambda},\boldsymbol{a}}(G).$$

In other words, the maps $\tau_{\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mu_{\boldsymbol{\lambda},\boldsymbol{a}}$ give an isomorphism of the regular representation of $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ and the $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$-module $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$.

Proof. The theorem follows from theorems 6.3, 6.7 and lemma 6.8. □

Remark. By lemma 4.10, the algebra $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is Frobenius. Therefore, its regular and coregular representations are isomorphic.

6.4. Isomorphism of algebras $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ and $\mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$

Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$ be a sequence of partitions with at most N parts such that $|\boldsymbol{\lambda}^{(s)}| = n_s$ for all $s = 1, \ldots, k$.

Consider the $\mathcal{B}$-module $\big(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\big)_{\boldsymbol{\lambda}}$. Denote $\big(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\big)_{\boldsymbol{\lambda}}$ by $\mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ and the Bethe algebra associated with $\big(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\big)_{\boldsymbol{\lambda}}$ by $\mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.

We begin with an observation. Let A be an associative unital algebra, and let L, M be A-modules such that L is isomorphic to a subquotient of M. Denote by A_L and A_M the images of A in $\mathrm{End}(L)$ and $\mathrm{End}(M)$, respectively, and by $\pi_L : A \to A_L, \pi_M : A \to A_M$ the corresponding epimorphisms. Then, there exists a unique epimorphism $\pi_{ML} : A_M \to A_L$ such that $\pi_L = \pi_{ML} \circ \pi_M$.

Applying this observation to the Bethe algebra $\mathcal{B}$ and $\mathcal{B}$-modules $\mathcal{M}_{\boldsymbol{\lambda}}, \mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}, \mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$, we get a chain of epimorphisms $\mathcal{B} \to \mathcal{B}_{\boldsymbol{\lambda}} \to \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$. In particular, each module over a smaller Bethe algebra is naturally a module over a bigger Bethe algebra.

For any element $F \in \mathcal{B}_{\boldsymbol{\lambda}}$, we denote by $\bar{F}$ the projection of F to the algebra $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$.

Let $C_1(u), \ldots, C_N(u)$ be the polynomials with coefficients in $\mathcal{B}_{\boldsymbol{\lambda}}$, defined in lemma 3.5. Introduce the elements $C_{ijs} \in \mathcal{B}_{\boldsymbol{\lambda}}$ for $i = 1, \ldots, N$, $j = 0, \ldots, n$, $s = 1, \ldots, k$, by the rule

$$\sum_{j=0}^{n} C_{ijs}(u - b_s)^j = C_i(u).$$

In addition, let $\bar{C}_{0js}$, $j = 0, \ldots, n$, $s = 1, \ldots, k$, be the numbers such that

$$\sum_{j=0}^{n} \bar{C}_{0js}(u - b_s)^j = \prod_{r=1}^{k}(u - b_r)^{n_r}.$$

Define the *indicial polynomial* $\bar{\chi}_s^{\mathcal{B}}(\alpha)$ *at* b_s by the formula

$$\bar{\chi}_s^{\mathcal{B}}(\alpha) = \sum_{i=0}^{N} \bar{C}_{i,n_s-i,s} \prod_{j=0}^{N-i-1} (\alpha - j).$$

It is a polynomial of degree N in the variable α with coefficients in $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$.

Let $I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ be the ideal in $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$ generated by the elements $\bar{C}_{ijs}$, $i = 1, \ldots, N$, $s = 1, \ldots, k$, $0 \leqslant j < n_s - i$, and the coefficients of the polynomials

$$\bar{\chi}_s^{\mathcal{B}}(\alpha) - \prod_{\substack{r=1\\ r\neq s}}^{k}(b_s - b_r)^{n_r} \prod_{l=1}^{N}\left(\alpha - \lambda_l^{(s)} - N + l\right), \qquad s = 1, \ldots, k. \tag{6.5}$$

Lemma 6.10. *The ideal* $I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *belongs to the kernel of the projection* $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.

Proof. The statement follows from lemma 3.4 and corollary 3.7. □

Hence, the projection $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ descends to an epimorphism

$$\pi_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} : \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} \big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \to \mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}, \tag{6.6}$$

which makes $\mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ into a $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} \big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module.

Denote $\ker\left(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\right) = \left\{v \in \mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}} \middle| I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} v = 0\right\}$. Clearly, $\ker\left(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\right)$ is a $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$-submodule of $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$.

Proposition 6.11. *The* $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}} \big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$*-modules* $\ker\left(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\right)$ *and* $\mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *are isomorphic.*

The proposition is proved in section 6.5.

Let $\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \subset \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ be the ideal defined in section 5.3. Clearly, the map $\tau_{\boldsymbol{\lambda},\boldsymbol{a}} : \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$ sends $\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ to $I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$. By lemma 5.7, the maps $\tau_{\boldsymbol{\lambda},\boldsymbol{a}}$ and $\pi_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ induce the homomorphism

$$\tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} : \mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \to \mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}.$$

By theorem 6.9, the map $\mu_{\boldsymbol{\lambda},\boldsymbol{a}} : \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$ sends $\mathrm{Ann}\left(\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\right) \subset \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ to $\ker\left(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\right)$. The vector spaces $\mathrm{Ann}\left(\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\right)$ and $(\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^*$ are isomorphic by corollary 5.8. Hence, proposition 6.11 yields that the map $\mu_{\boldsymbol{\lambda},\boldsymbol{a}}$ induces a bijective linear map

$$\mu_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} : (\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^* \to \mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}.$$

For any $F \in \mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$, denote by $F^* \in \mathrm{End}((\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^*)$ the operator, dual to the operator of multiplication by F on $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.

Theorem 6.12. *The map* $\tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *is an isomorphism of algebras. The maps* $\tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *and* $\mu_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *intertwine the action of the operators on* $(\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^*$*, dual to the multiplication operators on* $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$*, and the action of the Bethe algebra* $\mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *on* $\mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$*, that is, for any* $F \in \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$ *and* $G \in (\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})$*, we have*

$$\mu_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}(F^*G) = \tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}(F)\mu_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}(G).$$

In other words, the maps $\tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *and* $\mu_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *give an isomorphism of the coregular representation of* $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ *on the dual space* $(\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^*$ *and the* $\mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$*-module* $\mathcal{M}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$.

Proof. By lemma 5.7, the isomorphism $\tau_{\boldsymbol{\lambda},\boldsymbol{a}} : \mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}} \to \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$ induces the isomorphism

$$\tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} : \mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \to \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}\big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}.$$

so the maps $\tau_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ and $\mu_{\boldsymbol{\lambda},\boldsymbol{a}}$ give an isomorphism of the $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module $\mathrm{Ann}\big(\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big)$ and the $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}\big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module $\ker\big(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big)$ (see theorem 6.9).

By lemma 4.7, the $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module $\mathrm{Ann}\big(\bar{I}^{\mathcal{O}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big)$ is isomorphic to the coregular representation of $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ on the dual space $(\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}})^*$. In particular, it is faithful. Therefore, the $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}\big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module $\ker\big(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big)$ is faithful. By proposition 6.11, the $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}\big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$-module $M_{\boldsymbol{\lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, isomorphic to $\ker\big(I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}\big)$, is faithful too, which implies that the map $\pi_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} : \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}\big/ I^{\mathcal{B}}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}} \to \mathcal{B}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ is an isomorphism of algebras. The theorem follows. □

Remark. By lemma 5.3, the algebra $\mathcal{O}_{\Lambda,\boldsymbol{\lambda},\boldsymbol{b}}$ is Frobenius. Therefore, its coregular and regular representations are isomorphic.

6.5. *Proof of proposition 6.11*

We begin the proof with an elementary auxiliary lemma. Let M be a finite-dimensional vector space, $U \subset M$ be a subspace and $E \in \mathrm{End}(M)$.

Lemma 6.13. *Let $EM \subset U$, and the restriction of E to U is invertible in* $\mathrm{End}(U)$. *Then $EU = U$ and $M = U \oplus \ker E$.*

Let W_m be a Weyl module, see section 2.3, and $\boldsymbol{\mu}$ be a partition with at most N parts such that $|\boldsymbol{\mu}| = m$. Recall that W_m is a graded vector space, the grading of W_m is defined in lemma 2.2.

Given a homogeneous vector $w \in (W_m)^{\mathrm{sing}}_{\boldsymbol{\mu}}$, let $\mathcal{L}_w(b)$ be the $\mathfrak{gl}_N[t]$-submodule of $W_m(b)$ generated by the vector v. The space $\mathcal{L}_w(b)$ is graded. Denote by $\mathcal{L}^{=}_w(b)$ and $\mathcal{L}^{>}_w(b)$ the subspaces of $\mathcal{L}_w(b)$ spanned by homogeneous vectors of degree $\deg w$ and of degree strictly greater than $\deg w$, respectively. The subspace $\mathcal{L}^{=}_w(b)$ is a $\mathfrak{gl}_N$-submodule of $\mathcal{L}_w(b)$ isomorphic to the irreducible $\mathfrak{gl}_N$-module $L_{\boldsymbol{\mu}}$. The subspace $\mathcal{L}^{>}_w(b)$ is a $\mathfrak{gl}_N[t]$-submodule of $\mathcal{L}_w(b)$, and the $\mathfrak{gl}_N[t]$-module $\mathcal{L}_w(b)/\mathcal{L}^{>}_w(b)$ is isomorphic to the evaluation module $L_{\boldsymbol{\mu}}(b)$. If v has the largest degree possible for vectors in $(W_m)^{\mathrm{sing}}_{\boldsymbol{\mu}}$, then $\mathcal{L}^{>}_w(b)$, considered as a $\mathfrak{gl}_N$-module, does not contain $L_{\boldsymbol{\mu}}$.

For any $s = 1, \ldots, k$, pick up a homogeneous vector $w_s \in (W_{n_s})^{\mathrm{sing}}_{\boldsymbol{\lambda}^{(s)}}$ of the largest possible degree. Let $\mathcal{L}_{\boldsymbol{w}}(\boldsymbol{b})$ be the $\mathfrak{gl}_N[t]$-submodule of $\otimes_{s=1}^k W_{n_s}(b_s)$ generated by the vector $\otimes_{s=1}^k w_s$.

Denote by $\mathcal{L}^{=}_{\boldsymbol{w}}(\boldsymbol{b})$ and $\mathcal{L}^{>}_{\boldsymbol{w}}(\boldsymbol{b})$ the following subspaces of $\mathcal{L}_{\boldsymbol{w}}(\boldsymbol{b})$:

$$\mathcal{L}^{=}_{\boldsymbol{w}}(\boldsymbol{b}) = \otimes_{s=1}^k \mathcal{L}^{=}_{w_s}(b_s),$$

$$\mathcal{L}^{>}_{\boldsymbol{w}}(\boldsymbol{b}) = \sum_{s=1}^{k} \mathcal{L}_{w_1}(b_1) \otimes \cdots \otimes \mathcal{L}^{>}_{w_s}(b_s) \otimes \cdots \otimes \mathcal{L}_{w_k}(b_k).$$

The subspace $\mathcal{L}^{>}_{\boldsymbol{w}}(\boldsymbol{b})$ is a $\mathfrak{gl}_N[t]$-submodule of $\mathcal{L}_{\boldsymbol{w}}(\boldsymbol{b})$, and the $\mathfrak{gl}_N[t]$-module $\mathcal{L}_{\boldsymbol{w}}(\boldsymbol{b})/\mathcal{L}^{>}_{\boldsymbol{w}}(\boldsymbol{b})$ is isomorphic to the tensor product of evaluation modules $\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)$.

The space $\otimes_{s=1}^k W_{n_s}$ has the second $\mathfrak{gl}_N[t]$-module structure, denoted $\mathrm{gr}\big(\otimes_{s=1}^k W_{n_s}(b_s)\big)$, which was introduced at the end of section 2.3. The subspace $\mathcal{L}_{\boldsymbol{w}}(\boldsymbol{b})$ is a $\mathfrak{gl}_N[t]$-submodule of $\mathrm{gr}\big(\otimes_{s=1}^k W_{n_s}(b_s)\big)$, isomorphic to a direct sum of irreducible $\mathfrak{gl}_N[t]$-modules of the form $\otimes_{s=1}^k L_{\boldsymbol{\mu}^{(s)}}(b_s)$, where $|\boldsymbol{\mu}^{(s)}| = n_s, s = 1, \ldots, k$, see lemmas 2.5 and 2.6, and $(\boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(k)}) \neq (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$ for any term of the sum.

The subspace $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}} = \big(\otimes_{s=1}^k W_{n_s}(b_s)\big)_{\boldsymbol{\lambda}}$ is invariant under the action of the Bethe algebra $\mathcal{B} \subset U(\mathfrak{gl}_N[t])$. This makes it a $\mathcal{B}$-module, which we call the standard $\mathcal{B}$-module structure on $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$. The $\mathcal{B}$-module $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$ contains the submodules $\mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = (\mathcal{L}_{\boldsymbol{w}}(\boldsymbol{b}))_{\boldsymbol{\lambda}}$ and $\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \big(\mathcal{L}^{>}_{\boldsymbol{w}}(\boldsymbol{b})\big)_{\boldsymbol{\lambda}}$, and the subspace $\mathcal{M}^{\boldsymbol{w},=}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \big(\mathcal{L}^{=}_{\boldsymbol{w}}(\boldsymbol{b})\big)_{\boldsymbol{\lambda}}$. As vector spaces, $\mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \mathcal{M}^{\boldsymbol{w},=}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \oplus \mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. The $\mathcal{B}$-modules $\mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big/\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ and $\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ are isomorphic.

The space $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$ has another $\mathcal{B}$-module structure, inherited from the $\mathfrak{gl}_N[t]$-module structure of $\mathrm{gr}\big(\otimes_{s=1}^{k} W_{n_s}(b_s)\big)$. We denote the new structure $\mathrm{gr}\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$. The subspaces $\mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}, \mathcal{M}^{\boldsymbol{w},=}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}, \mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ are $\mathcal{B}$-submodules of the $\mathcal{B}$-module $\mathrm{gr}\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$. The submodule $\mathcal{M}^{\boldsymbol{w},=}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \subset \mathrm{gr}\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is isomorphic to the $\mathcal{B}$-module $\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, and the submodule $\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \subset \mathrm{gr}\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is isomorphic to a direct sum of $\mathcal{B}$-modules of the form $\mathcal{M}_{\boldsymbol{M},\boldsymbol{\lambda},\boldsymbol{b}}$, where $\boldsymbol{M} = (\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(k)})$, $|\boldsymbol{\mu}^{(s)}| = n_s$, $s = 1, \dots, k$, and $\mathbf{M} \neq \boldsymbol{\Lambda}$ for any term of the sum.

In the picture described above, we can regard all $\mathcal{B}$-modules involved as $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$-modules.

For any $F \in \mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$, we denote by $\mathrm{gr} F \in \mathrm{End}(\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}})$ the linear operator corresponding to the action of F on $\mathrm{gr}\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$. The map $F \mapsto \mathrm{gr}\, F$ is an algebra homomorphism.

Let complex numbers $c_1, \dots, c_k, \alpha_1, \dots, \alpha_k$ be such that

$$\sum_{s=1}^{k} c_s \left(\prod_{i=1}^{N} \big(\alpha_s - \mu_i^{(s)} - N + i\big) - \prod_{i=1}^{N} \big(\alpha_s - \lambda_i^{(s)} - N + i\big) \right) \neq 0,$$

for any sequence of partitions $(\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(k)}) \neq (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$. Introduce

$$E = \sum_{s=1}^{k} c_s \left(\bar{\chi}_s^{\mathcal{B}}(\alpha_s) - \prod_{\substack{r=1\\ r\neq s}}^{k} (b_s - b_r)^{n_r} \prod_{i=1}^{N} \big(\alpha_s - \lambda_i^{(s)} - N + i\big) \right) \in I^{\mathcal{B}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}},$$

where $\bar{\chi}_s^{\mathcal{B}}$ is the indicial polynomial (6.5). With respect to the standard $\mathcal{B}$-module structure on $\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{a}}$, we have $E\mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \subset \mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.

Lemma 6.14. *The restriction of E to $\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is invertible in* $\mathrm{End}\big(\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big)$.

Proof. Lemma 6.10 implies that the projection of E to $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ equals zero, and the projection of E to $\mathcal{B}_{\mathbf{M},\boldsymbol{\lambda},\boldsymbol{b}}$ with $\mathbf{M} \neq \boldsymbol{\Lambda}$ is invertible. This means that the restriction of the operator $\mathrm{gr}E$ to $\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is invertible in $\mathrm{End}\big(\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big)$. Therefore, the restriction of E to $\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is invertible in $\mathrm{End}\big(\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big)$. □

Denote $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E = \ker E \cap \mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. By lemma 6.13, the canonical projection

$$\mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \to \mathcal{M}^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big/\mathcal{M}^{\boldsymbol{w},>}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \simeq \mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$$

induces an isomorphism $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E \to \mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ of vector spaces. Since the algebra $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$ is commutative, the subspace $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E$ is a $\mathcal{B}$-submodule and the map $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E \to \mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is an isomorphism of $\mathcal{B}_{\boldsymbol{\lambda},\boldsymbol{a}}$-modules.

Lemma 6.10 implies that elements of the ideal $I^{\mathcal{B}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ act on $\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ by zero. Hence, they act by zero on $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E$, that is, $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E \subset \ker\big(I^{\mathcal{B}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big)$. On the other hand, we have

$$\dim \ker\big(I^{\mathcal{B}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big) = \dim \mathrm{Ann}\big(\bar{I}^{\mathcal{O}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big) = \dim \mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \dim \mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} = \dim \ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E,$$

see theorem 6.9, corollary 5.8 and formula (5.4), which yields $\ker^{\boldsymbol{w}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} E = \ker\big(I^{\mathcal{B}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}\big)$. Proposition 6.11 is proved.

Remark. Note that formula (5.4) is the key ingredient of the proof.

7. Applications

7.1. Action of the Bethe algebra in a tensor product of evaluation modules

In this section, we summarize the obtained results in a way independent from the main part of the paper. For convenience, we recall some definitions.

Let $\boldsymbol{K} = (K_1, \ldots, K_N)$ be a sequence of distinct complex numbers. The Bethe algebra $\mathcal{B}$ is a commutative subalgebra of $U(\mathfrak{gl}_N[t])$, defined in section 3.1 with the help of this sequence. It is generated by the elements B_{ij}, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$, given by formula (3.2). The Bethe algebra depends on the choice of $\boldsymbol{K}$. In the remainder of the paper we will denote this algebra by $\mathcal{B}_{\boldsymbol{K}}$.

If M is a $\mathcal{B}_{\boldsymbol{K}}$-module and $\xi : \mathcal{B}_{\boldsymbol{K}} \to \mathbb{C}$ a homomorphism, then the eigenspace of the $\mathcal{B}_{\boldsymbol{K}}$-action on M corresponding to ξ is defined as $\bigcap_{F\in\mathcal{B}_{\boldsymbol{K}}} \ker(F|_M - \xi(F))$ and the generalized eigenspace of the $\mathcal{B}_{\boldsymbol{K}}$-action on M corresponding to ξ is defined as $\bigcap_{F\in\mathcal{B}_{\boldsymbol{K}}} \left(\bigcup_{m=1}^{\infty} \ker(F|_M - \xi(F))^m\right)$.

For a partition $\boldsymbol{\lambda}$ with at most N parts, let $L_{\boldsymbol{\lambda}}$ be the irreducible finite-dimensional $\mathfrak{gl}_N$-module of highest weight $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}$ be partitions with at most N parts, $b_1, \ldots, b_k$ distinct complex numbers. We are interested in the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on the tensor product $\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)$ of evaluation $\mathfrak{gl}_N[t]$-modules.

Since $\mathcal{B}_{\boldsymbol{K}}$ commutes with the subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N[t])$, the action of $\mathcal{B}_{\boldsymbol{K}}$ preserves the weight subspaces of $\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)$.

Denote $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$. Given a partition $\boldsymbol{\lambda}$ with at most N parts such that $|\boldsymbol{\lambda}| = \sum_{s=1}^{k} |\boldsymbol{\lambda}^{(s)}|$, let $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$ be the set of all monic differential operators of order N,

$$\mathcal{D} = \partial^N + \sum_{i=1}^{N} h_i^{\mathcal{D}}(u)\partial^{N-i}, \tag{7.1}$$

where $\partial = d/du$, with the following properties:

(a) The singular points of $\mathcal{D}$ are at $b_1, \ldots, b_k$ and ∞ only.
(b) The exponents of $\mathcal{D}$ at b_s, $s = 1, \ldots, k$, are equal to $\lambda_N^{(s)}, \lambda_{N-1}^{(s)} + 1, \ldots, \lambda_1^{(s)} + N - 1$.
(c) The kernel of $\mathcal{D}$ is generated by quasi-exponentials of the form

$$g_i(u) = \mathrm{e}^{K_i u}(u^{\lambda_i} + g_{i1}u^{\lambda_i - 1} + \cdots + g_{i\lambda_i}), \qquad i = 1, \ldots, N,$$

where g_{ij} are suitable complex numbers.

A differential operator $\mathcal{D}$ belongs to the set $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$ if and only if the kernel of $\mathcal{D}$ is a point of the intersection $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ (see lemma 5.1).

Denote $n_s = |\boldsymbol{\lambda}^{(s)}|$, $s = 1, \ldots, k$ and $n = \sum_{s=1}^{k} n_s$.

Theorem 7.1. *The action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on $\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)$ has the following properties:*

(i) For every $i = 1, \ldots, N$, the action of the series $B_i(u)$ is given by the power series expansion in u^{-1} of a rational function of the form $A_i(u)\prod_{s=1}^{k}(u - b_s)^{-n_s}$, where $A_i(u)$ is a polynomial of degree n with coefficients in $\mathrm{End}\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)$.

(ii) The image of $\mathcal{B}_{\boldsymbol{K}}$ in $\mathrm{End}\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)$ *is a maximal commutative subalgebra of dimension* $\dim \otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}$.

(iii) Each eigenspace of the action of $\mathcal{B}_{\boldsymbol{K}}$ is one-dimensional.

(iv) Each generalized eigenspace of the action of $\mathcal{B}_{\boldsymbol{K}}$ is generated over $\mathcal{B}_{\boldsymbol{K}}$ by one vector.

(v) *The eigenspaces of the action of* $\mathcal{B}_{\boldsymbol{K}}$ *on* $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ *are in a one-to-one correspondence with differential operators from* $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$. *Moreover, if* $\mathcal{D}$ *is the differential operator, corresponding to an eigenspace, then the coefficients of the series* $h_i^{\mathcal{D}}(u)$ *are the eigenvalues of the action of the respective coefficients of the series* $B_i(u)$.

(vi) *The eigenspaces of the action of* $\mathcal{B}_{\boldsymbol{K}}$ *on* $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ *are in a one-to-one correspondence with points of the intersection* $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, *given by* (5.1).

Proof. The first property follows from corollary 3.7. The other properties follow from theorem 6.12, lemma 5.1 and standard facts about the coregular representations of Frobenius algebras given in section 4.3. □

Corollary 7.2. *The following three statements are equivalent:*

(i) *The action of the Bethe algebra* $\mathcal{B}_{\boldsymbol{K}}$ *on* $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ *is diagonalizable.*

(ii) *The set* $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$ *consists of* $\dim\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$ *distinct points.*

(iii) *The set* $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ *consists of* $\dim\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$ *distinct points.*

The intersection $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is *transversal* if the scheme-theoretic intersection $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is a direct sum of one-dimensional algebras.

Corollary 7.3. *The action of the Bethe algebra* $\mathcal{B}_{\boldsymbol{K}}$ *on* $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ *is diagonalizable, if and only the* $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ *is transversal.*

Proof. The algebra $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is a direct sums of local algebras, each local summand corresponding to a point of the set $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. Therefore, the intersection $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is transversal if and only if the dimension of $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ equals the cardinality of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. Corollary 7.2 completes the proof. □

Corollary 7.4. *Let* $K_1, \ldots, K_N$ *be distinct real numbers. Let* $b_1, \ldots, b_k$ *be distinct real numbers. Then*

(i) *The set* $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$ *consists of* $\dim\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$ *distinct points.*

(ii) *The intersection* $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ *consists of* $\dim\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}\right)_{\boldsymbol{\lambda}}$ *distinct points and is transversal.*

Proof. If $K_1, \ldots, K_N$ are distinct real numbers and $b_1, \ldots, b_k$ are distinct real numbers, then the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on $\left(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ is diagonalizable, see [MTV1] (cf [MTV2]). □

7.2. Action of the Bethe algebra in Weyl modules

Results similar to theorem 7.1 and corollary 7.2 hold for the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^{k} W_{n_s}(b_s)$, the Weyl module associated with $\boldsymbol{n} = (n_1, \ldots, n_k)$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$, defined in section 2.3. The action of $\mathcal{B}_{\boldsymbol{K}}$ preserves the weight subspaces of $\otimes_{s=1}^{k} W_{n_s}(b_s)$.

Recall that V denotes the irreducible $\mathfrak{gl}_N$-module of highest weight $(1, 0, \ldots, 0)$, which is the vector representation of $\mathfrak{gl}_N$.

Denote by $\Delta_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$ the set of all monic differential operators $\mathcal{D}$ of order N with the following properties.

(a) The kernel of $\mathcal{D}$ is generated by quasi-exponentials of the form

$$g_i(u) = \mathrm{e}^{K_i u}(u^{\lambda_i} + g_{i1}u^{\lambda_i - 1} + \cdots + g_{i\lambda_i}), \qquad i = 1, \ldots, N,$$

where $\lambda_1 + \cdots + \lambda_N = n$ and g_{ij} are suitable complex numbers.

(b) The first coefficient $h_1^{\mathcal{D}}(u)$ of $\mathcal{D}$, see (7.1), equals $\sum_{i=1}^N K_i + \sum_{s=1}^k n_s(b_s - u)^{-1}$.

If $\mathcal{D} \in \Delta_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$, then $\mathcal{D}$ is a differential operator with singular points at $b_1, \ldots, b_k$ and ∞ only.

Denote by $\Omega_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$ the set of all N-dimensional spaces of quasi-exponentials with a basis of the form

$$g_i(u) = \mathrm{e}^{K_i u}(u^{\lambda_i} + g_{i1}u^{\lambda_i - 1} + \cdots + g_{i\lambda_i}), \qquad i = 1, \ldots, N,$$

and such that

$$\mathrm{Wr}(g_1(u), \ldots, g_N(u)) = \mathrm{e}^{\sum_{i=1}^N K_i u} \prod_{1 \leqslant i < j \leqslant N} (K_j - K_i) \prod_{s=1}^k (u - b_s)^{n_s}.$$

A differential operator $\mathcal{D}$ belongs to the set $\Delta_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$ if and only if the kernel of $\mathcal{D}$ belongs to the set $\Omega_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$.

Theorem 7.5. *The action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on $\otimes_{s=1}^k W_{n_s}(b_s)$ has the following properties.*

(i) For every $i = 1, \ldots, N$, the action of the series $B_i(u)$ is given by the power series expansion in u^{-1} of a rational function of the form $A_i(u)\prod_{s=1}^k (u - b_s)^{-n_s}$, where $A_i(u)$ is a polynomial of degree n with coefficients in $\mathrm{End}\left(\otimes_{s=1}^k W_{n_s}\right)$.

(ii) The image of $\mathcal{B}_{\boldsymbol{K}}$ in $\mathrm{End}\left(\otimes_{s=1}^k W_{n_s}\right)$ *is a maximal commutative subalgebra of dimension* $\dim V^{\otimes n}$.

(iii) Each eigenspace of the action of $\mathcal{B}_{\boldsymbol{K}}$ is one-dimensional.

(iv) Each generalized eigenspace of the action of $\mathcal{B}_{\boldsymbol{K}}$ is generated over $\mathcal{B}_{\boldsymbol{K}}$ by one vector.

(v) The eigenspaces of the action of $\mathcal{B}_{\boldsymbol{K}}$ on $\otimes_{s=1}^k W_{n_s}(b_s)$ are in a one-to-one correspondence with differential operators from $\Delta_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$. Moreover, if $\mathcal{D}$ is the differential operator, corresponding to an eigenspace, then the coefficients of the series $h_i^{\mathcal{D}}(u)$ are the eigenvalues of the action of the respective coefficients of the series $B_i(u)$.

(vi) The eigenspaces of the action of $\mathcal{B}_{\boldsymbol{K}}$ on $\otimes_{s=1}^k W_{n_s}(b_s)$ are in a one-to-one correspondence with spaces of polynomials from $\mathrm{Wr}_{\boldsymbol{n},\boldsymbol{b}}^{-1}$.

Proof. The first property follows from lemmas 2.14 and 3.5. The other properties follow from theorem 6.9, formulae (4.4) and (6.2), and standard facts about the regular representations of Frobenius algebras given in section 4.3. □

Corollary 7.6. *The following three statements are equivalent.*

(i) The action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$ on $\otimes_{s=1}^k W_{n_s}(b_s)$ is diagonalizable.

(ii) The set $\Delta_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$ consists of $\dim V^{\otimes n}$ *distinct points.*

(iii) The set $\Omega_{\boldsymbol{n},\boldsymbol{b},\boldsymbol{K}}$ consists of $\dim V^{\otimes n}$ *distinct points.*

8. Completeness of Bethe ansatz

8.1. Generic points of $\bar{\Omega}_{\boldsymbol{\lambda}}$

Let $\bar{\Omega}_{\boldsymbol{\lambda}}$ be the affine $(n+N)$-dimensional space with coordinates $g_{ij}, i = 1, \ldots, N, j = 1, \ldots, \lambda_i$ and $k_1, \ldots, k_N$. We identify points $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$ with N-dimensional complex vector spaces generated by quasi-exponentials

$$g_i(u, Y) = \mathrm{e}^{k_i(Y)_i u}(u^{\lambda_i} + g_{i1}(Y)u^{\lambda_i - 1} + \cdots + g_{i\lambda_i}(Y)), \qquad i = 1, \ldots, N. \tag{8.1}$$

Let $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$ be a point with distinct coordinates $k_1(Y), \ldots, k_N(Y)$. Denote by $\mathcal{B}_Y \subset U(\mathfrak{gl}_N[t])$ the Bethe algebra constructed in section 3.1 with the help of the sequence $\boldsymbol{K} = (K_1, \ldots, K_N)$ where $K_i = k_i(Y)$ for all i.

For $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$, introduce the polynomials $\{y_0(u), y_1(u), \ldots, y_{N-1}(u)\}$, by the formula

$$y_a(u)\,\mathrm{e}^{\sum_{i=a+1}^{N} k_i(Y)u} \prod_{a<i<j\leqslant N} (k_i(Y) - k_j(Y)) = \mathrm{Wr}(g_{a+1}(u, Y), \ldots, g_N(u, Y)),$$

for $a = 0, \ldots, N$. Set

$$l_a = \sum_{b=a+1}^{N} \lambda_b, \qquad a = 0, \ldots, N. \tag{8.2}$$

Clearly, $l_0 = |\boldsymbol{\lambda}|$ and $l_N = 0$.

For each $a = 0, \ldots, N-1$, the polynomial $y_a(u)$ is a monic polynomial of degree l_a. Denote $t_1^{(a)}, \ldots, t_{l_a}^{(a)}$ the roots of the polynomial $y_a(u)$, and put

$$\boldsymbol{t}_Y = \big(t_1^{(0)}, \ldots, t_{l_0}^{(0)}, \ldots, t_1^{(N-1)}, \ldots, t_{l_{N-1}}^{(N-1)}\big). \tag{8.3}$$

We say that $\boldsymbol{t}_Y$ are the *root coordinates* of Y.

We say that $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$ is *generic* if all roots of the polynomials $y_0(u), y_1(u), \ldots, y_{N-1}(u)$ are simple and for each $a = 1, \ldots, N-1$, the polynomials $y_{a-1}(u)$ and $y_a(u)$ do not have common roots.

If Y is generic, then the root coordinates $\boldsymbol{t}_Y$ satisfy the Bethe ansatz equations [MV1] (cf [MTV4]),

$$\sum_{j'=1}^{l_{a-1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a-1)}} - \sum_{\substack{j'=1\\ j'\neq j}}^{l_a} \frac{2}{t_j^{(a)} - t_{j'}^{(a)}} + \sum_{j'=1}^{l_{a+1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a+1)}} = K_{a+1} - K_a.$$

Here the equations are labeled by $a = 1, \ldots, N-1$, $j = 1, \ldots, l_a$.

Conversely, if $\boldsymbol{t} = \big(t_1^{(0)}, \ldots, t_{l_0}^{(0)}, \ldots, t_1^{(N-1)}, \ldots, t_{l_{N-1}}^{(N-1)}\big)$ satisfy the Bethe ansatz equations, then there exists a unique $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$ such that Y is generic and $\boldsymbol{t}$ are its root coordinates. This Y is determined by the following construction, see [MV1] (cf [MTV4]). Set

$$\chi^a(u, \boldsymbol{t}) = K_a + \sum_{j=1}^{l_{a-1}} \frac{1}{u - t_j^{(a-1)}} - \sum_{i=1}^{l_a} \frac{1}{u - t_j^{(a)}}, \qquad a = 1, \ldots, N.$$

Then the monic differential operator $\mathcal{D}_Y$ with kernel Y is given by the formula

$$\mathcal{D}_Y = (\partial - \chi^1(u, \boldsymbol{t}))\ldots(\partial - \chi^N(u, \boldsymbol{t})).$$

Clearly, the operator $\mathcal{D}_Y$ determines Y.

Lemma 8.1. *Generic points form a Zariski open subset of* $\bar{\Omega}_{\boldsymbol{\lambda}}$.

The lemma follows from theorem 10.5.1 in [MTV4].

8.2. Universal weight function

Let $\boldsymbol{\lambda}$ be a partition with at most N parts. Let $l_0, \ldots, l_N$ be the numbers defined in (8.2). Denote $n = l_0, l = l_1 + \cdots + l_{N-1}$ and $\boldsymbol{l} = (l_1, \ldots, l_{N-1})$.

Consider the weight subspace $(V^{\otimes n})_{\boldsymbol{\lambda}}$ of the nth tensor power of the vector representation of $\mathfrak{gl}_N$ and the space $\mathbb{C}^{l+n}$ with coordinates $\boldsymbol{t} = \big(t_1^{(0)}, \ldots, t_{l_0}^{(0)}, \ldots, t_1^{(N-1)}, \ldots, t_{l_{N-1}}^{(N-1)}\big)$.

In this section we remind the construction of a rational map $\omega : \mathbb{C}^{l+n} \to (V^{\otimes n})_{\boldsymbol{\lambda}}$, called the *universal weight function* (see [SV]).

A basis of $V^{\otimes n}$ is formed by the vectors

$$e_J v = e_{j_1,1}v_+ \otimes \cdots \otimes e_{j_n,1}v_+,$$

where $J = (j_1, \ldots, j_n)$ and $1 \leqslant j_s \leqslant N$ for $s = 1, \ldots, N$. A basis of $(V^{\otimes n})_{\boldsymbol{\lambda}}$ is formed by the vectors $e_J v$ such that $\#\{s | j_s > i\} = l_i$ for every $i = 1, \ldots, N-1$. Such a J will be called $\boldsymbol{l}$-admissible.

The universal weight function has the form

$$\omega(\boldsymbol{t}) = \sum_J \omega_J(\boldsymbol{t}) e_J v,$$

where the sum is over the set of all $\boldsymbol{l}$-admissible J, and the function $\omega_J(\boldsymbol{t})$ is defined below.

For an admissible J, define $S(J) = \{s | j_s > 1\}$, and for $i = 1, \ldots, N-1$, define

$$S_i(J) = \{s | 1 \leqslant s \leqslant n, 1 \leqslant i < j_s\}.$$

Then $|S_i(J)| = l_i$.

Let $B(J)$ be the set of sequences $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{N-1})$ of bijections $\beta_i : S_i(J) \to \{1, \ldots, l_i\}, i = 1, \ldots, N-1$. Then $|B(J)| = \prod_{a=1}^{N-1} l_a!$.

For $s \in S(J)$ and $\boldsymbol{\beta} \in B(J)$, introduce the rational function

$$\omega_{s,\boldsymbol{\beta}}(\boldsymbol{t}) = \frac{1}{t^{(1)}_{\beta_1(s)} - t^{(0)}_s} \prod_{i=2}^{j_1 - 1} \frac{1}{t^{(i)}_{\beta_i(s)} - t^{(i-1)}_{\beta_{i-1}(s)}}$$

and define

$$\omega_J(\boldsymbol{t}) = \sum_{\boldsymbol{\beta} \in B(J)} \prod_{s \in S(J)} \omega_{s,\boldsymbol{\beta}}.$$

Example 8.2. Let $n = 2$ and $\boldsymbol{l} = (1, 1, 0, \ldots, 0)$. Then

$$\omega(\boldsymbol{t}) = \frac{1}{\left(t^{(2)}_1 - t^{(1)}_1\right)\left(t^{(1)}_1 - t^{(0)}_1\right)} e_{3,1}v_+ \otimes v_+ + \frac{1}{\left(t^{(2)}_1 - t^{(1)}_1\right)\left(t^{(1)}_1 - t^{(0)}_2\right)} v_+ \otimes e_{3,1}v_+.$$

Theorem 8.2. *Let $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$ be a generic point with root coordinates $\boldsymbol{t}_Y$. Consider the value $\omega(\boldsymbol{t}_Y)$ of the universal weight function $\omega : \mathbb{C}^{l+n} \to (V^{\otimes n})_{\boldsymbol{\lambda}}$ at $\boldsymbol{t}_Y$. Consider $V^{\otimes n}$ as the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^n V\left(t^{(0)}_s\right)$. Consider the Bethe algebra $\mathcal{B}_Y \subset U(\mathfrak{gl}_N[t])$. Then the vector $\omega(\boldsymbol{t}_Y)$ is an eigenvector of the Bethe algebra $\mathcal{B}_Y$, acting on $\otimes_{s=1}^n V\left(t^{(0)}_s\right)$. Moreover, $\mathcal{D}^{\mathcal{B}_Y}_{\omega(\boldsymbol{t}_Y)} = \mathcal{D}_Y$, where $\mathcal{D}^{\mathcal{B}_Y}_{\omega(\boldsymbol{t}_Y)}$ and $\mathcal{D}_Y$ are the differential operators associated with the eigenvector $\omega(\boldsymbol{t}_Y)$ and the point $Y \in \bar{\Omega}_{\boldsymbol{\lambda}}$, respectively.*

The theorem is proved in [MTV1].

8.3. Epimorphism $F_{\boldsymbol{\lambda}}$

Let $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\lambda}$ be partitions with at most N parts such that $|\boldsymbol{\lambda}| = \sum_{s=1}^k |\boldsymbol{\lambda}^{(s)}|$, and $b_1, \ldots, b_k$ distinct complex numbers. Denote $n = |\boldsymbol{\lambda}|$ and $n_s = |\boldsymbol{\lambda}^{(s)}|$, $s = 1, \ldots, k$.

For $s = 1, \ldots, k$, let $F_s : V^{\otimes n_s} \to L_{\boldsymbol{\lambda}^{(s)}}$ be an epimorphism of $\mathfrak{gl}_N$-modules. Then

$$F_1 \otimes \cdots \otimes F_k : \otimes_{s=1}^k V(b_s)^{\otimes n_s} \to \otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s) \tag{8.4}$$

is an epimorphism of $\mathfrak{gl}_N[t]$-module, which induces an epimorphism of $\mathcal{B}_Y$-modules

$$F : \left(\otimes_{s=1}^k V(b_s)^{\otimes n_s}\right)_{\boldsymbol{\lambda}} \to \left(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}},$$

for any Y with distinct coordinates $k_1(Y), \ldots, k_N(Y)$.

8.4. Construction of an eigenvector from a differential operator

Let $\mathcal{D}^0$ be an element of $\Delta_{\mathbf{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$. Let Y^0 be the kernel of $\mathcal{D}^0$. Then Y^0 is a point of the cell $\bar{\Omega}_{\boldsymbol{\lambda}}$ and $K_i = k_i(Y^0)$ for all i. In particular, we have $\mathcal{B}_{\boldsymbol{K}} = \mathcal{B}_{Y^0}$.

Choose a germ of an algebraic curve $Y(\epsilon)$ in $\bar{\Omega}_{\boldsymbol{\lambda}}$ such that $Y(0) = Y^0$ and $Y(\epsilon)$ are generic points of $\bar{\Omega}_{\boldsymbol{\lambda}}$ for all nonzero ϵ. Let $\boldsymbol{t}(\epsilon)$ be the root coordinates of $Y(\epsilon)$. The algebraic functions $t_1^{(0)}(\epsilon), \dots, t_n^{(0)}(\epsilon)$ are determined up to permutation. Order them in such a way that the first n_1 of them tend to b_1 as $\epsilon \to 0$, the next n_2 coordinates tend to b_2 and so on until the last n_k coordinates tend to b_k.

For every nonzero ϵ, the vector $v(\epsilon) = \omega(\boldsymbol{t}(\epsilon))$ belongs to $(V^{\otimes n})_{\boldsymbol{\lambda}}$. This vector is an eigenvector of the Bethe algebra $\mathcal{B}_{Y(\epsilon)}$, acting on $(\otimes_{s=1}^n V(t_s^{(0)}(\epsilon)))_{\boldsymbol{\lambda}}$, and we have $\mathcal{D}_{v(\epsilon)}^{\mathcal{B}_{Y(\epsilon)}} = \mathcal{D}_{Y(\epsilon)}$ (see theorem 8.2).

The vector $v(\epsilon)$ algebraically depends on ϵ. Let $v(\epsilon) = v_0\epsilon^{a_0} + v_1\epsilon^{a_1} + \cdots$ be its Puiseux expansion, where v_0 is the leading nonzero coefficient.

Theorem 8.3. *For a generic choice of the maps $F_1, \dots, F_k$, the vector $F(v_0)$ is nonzero. Moreover, $F(v_0)$ is an eigenvector of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$, acting on $\left(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$, and $\mathcal{D}_{F(v_0)}^{\mathcal{B}_{K}} = \mathcal{D}^0$.*

Proof. For any generator $B_{ij} \in \mathcal{B}_{Y(\epsilon)}$, the action of B_{ij} on the $U(\mathfrak{gl}_N[t])$-module $\otimes_{s=1}^n V(t_s^{(0)}(\epsilon))$ determines an element of $\mathrm{End}(V^{\otimes n})$, algebraically depending on ϵ. Since for every nonzero ϵ, the vector $v(\epsilon)$ is an eigenvector of $\mathcal{B}_{Y(\epsilon)}$, acting on $(\otimes_{s=1}^n V(t_s^{(0)}(\epsilon)))_{\boldsymbol{\lambda}}$, and $\mathcal{D}_{v(\epsilon)}^{\mathcal{B}} = \mathcal{D}_{Y(\epsilon)}$, the vector v_0 is an eigenvector of $\mathcal{B}_{Y(0)} = \mathcal{B}_{\boldsymbol{K}}$, acting on $\left(\otimes_{s=1}^k V(b_s)^{\otimes n_s}\right)_{\boldsymbol{\lambda}}$, and $\mathcal{D}_{v_0}^{\mathcal{B}_K} = \mathcal{D}^0$.

The $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^k V(b_s)^{\otimes n_s}$ is a direct sum of irreducible $\mathfrak{gl}_N[t]$-modules of the form $\otimes_{s=1}^k L_{\boldsymbol{\mu}^{(s)}}(b_s)$, where $|\boldsymbol{\mu}^{(s)}| = n_s$, $s = 1, \dots, k$. Since $\mathcal{D}^0 \in \Delta_{\mathbf{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$, the vector v_0 belongs to the component of type $\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)$. Therefore, for generic choice of the maps $F_1, \dots, F_k$, the vector $F(v_0)$ is nonzero.

Since the map $F_1 \otimes \cdots \otimes F_k$, see (8.4), is a homomorphism of $\mathfrak{gl}_N[t]$-modules, the vector $F(v_0)$ is an eigenvector of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$, acting on $\left(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$, and $\mathcal{D}_{F(v_0)}^{\mathcal{B}_K} = \mathcal{D}^0$. □

Given $\mathcal{D} \in \Delta_{\mathbf{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$, denote by $w(\mathcal{D})$ the vector $F(v_0) \in \left(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ constructed from $\mathcal{D}$ in section 8.4. The vector $w(\mathcal{D})$ is defined up to multiplication by a nonzero number. The assignment $\mathcal{D} \mapsto w(\mathcal{D})$ gives the correspondence, which is inverse to the correspondence $v \mapsto \mathcal{D}_v^{\mathcal{B}}$ in part (v) of theorem 7.1.

8.5. Completeness of Bethe ansatz for $\mathfrak{gl}_N$ Gaudin model

The construction of the vector $w(\mathcal{D}) \in \left(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}}$ from a differential operator $\mathcal{D} \in \Delta_{\mathbf{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}$ can be viewed as a (generalized) Bethe ansatz construction for the $\mathfrak{gl}_N$ Gaudin model, cf the Bethe ansatz constructions in [Ba, RV, MV1, MV2].

Theorem 8.4. *If $b_1, \dots, b_k$ are distinct real numbers and $K_1, \dots, K_N$ are distinct real numbers, then the collection of vectors*

$$\left\{w(\mathcal{D}) \in \left(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)\right)_{\boldsymbol{\lambda}} \middle| \mathcal{D} \in \Delta_{\mathbf{\Lambda},\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{K}}\right\}$$

is an eigenbasis of the action of the Bethe algebra $\mathcal{B}_{\boldsymbol{K}}$.

The theorem follows from theorem 7.1 and corollaries 7.2 and 7.4.

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References

[Ba] Babujian H M 1993 Off-shell bethe ansatz equation and N-point correlators in the $SU(2)$ WZNW theory *J. Phys. A: Math. Gen.* **26** 6981–90

[CG] Chari V and Greenstein J 2007 Current algebras, highest weight categories and quivers *Adv. Math.* **216** 811–40

[CL] Chari V and Loktev S 2006 Weyl, Fusion and Demazure modules for the current algebra of sl_{r+1} *Adv. Math.* **207** 928–60

[CP] Chari V and Pressley A 2001 Weyl modules for classical and quantum affine algebras *Represent. Theory* **5** 191–223 (electronic)

[CT] Chervov A and Talalaev D 2006 Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence *Preprint* hep-th/0604128 (1–54)

[FFR] Feigin B, Frenkel E and Rybnikov L 2007 Opers with irregular singularity and spectra of the shift of argument subalgebra *Preprint* arXiv:0712.1183 (1–19)

[HU] Howe R and Umeda T 1991 The Capelli identity, the double commutant theorem, and multi-plicity-free actions *Math. Ann.* **290** 565–619

[K] Kedem R 2004 Fusion products of $\mathfrak{sl}_N$ symmetric power representations and Kostka polynomials *Quantum Theory and Symmetries* (Hackensack, NJ: World Scientific) pp 88–93

[M] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Oxford University Press)

[MNO] Molev A, Nazarov M and Olshanski G 1996 Yangians and classical Lie algebras *Russ. Math. Surv.* **51** 205–82

[MTV1] Mukhin E, Tarasov V and Varchenko A 2006 Bethe eigenvectors of higher transfer matrices *J. Stat. Mech.* P08002, 1–44

[MTV2] Mukhin E, Tarasov V and Varchenko A 2005 The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz *Preprint* math.AG/0512299 (1–18)

[MTV3] Mukhin E, Tarasov V and Varchenko A 2006 A generalization of the Capelli identity *Preprint* math.QA/0610799 (1–14)

[MTV4] Mukhin E, Tarasov V and Varchenko A 2007 Generating operator of *XXX* or Gaudin transfer matrices has quasi-exponential Kernel *SIGMA* **3 060** 1–31 (electronic)

[MTV5] Mukhin E, Tarasov V and Varchenko A 2007 On reality property of Wronski maps *Preprint* arXiv:0710.5856 (1–15)

[MTV6] Mukhin E, Tarasov V and Varchenko A 2007 Schubert calculus and representations of general linear group *Preprint* arXiv:0711.4079 (1–32)

[MTV7] Mukhin E, Tarasov V and Varchenko A 2007 On separation of variables and completeness of the Bethe ansatz for quantum $\mathfrak{gl}_N$ Gaudin model *Preprint* arXiv:0712.0981 (1–9)

[MTV8] Mukhin E, Tarasov V and Varchenko A 2006 Bispectral and $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ dualities *Func. Anal. Other Math.* **1** 55–80

[MV1] Mukhin E and Varchenko A 2006 Spaces of quasi-polynomials and the Bethe Ansatz *Preprint* math.QA/0604048 (1–29)

[MV2] Mukhin E and Varchenko A 2005 Norm of a Bethe vector and the Hessian of the master function *Compos. Math.* **141** 1012–28

[RV] Reshetikhin N and Varchenko A 1995 Quasiclassical asymptotics of solutions to the KZ equations *Geometry, Topology and Physics for R. Bott (Intern. Press)* pp 293–322

[Sk] Sklyanin E 1989 Separation of variables in the Gaudin model *J. Sov. Math.* **47** 2473–88

[SV] Schechtman V and Varchenko A 1991 Arrangements of hyperplanes and Lie algebra homology *Invent. Math.* **106** 139–94

[T] Talalaev D 2004 Quantization of the Gaudin system *Preprint* hep-th/0404153 (1–19)